Helical Attractors on Contact 3-Manifolds: Basin Geometry, Whitney Singularities, and the Three-Disk Plasma Engine

Pablo Nogueira Grossi · G6 LLC, Newark, New Jersey, USA · grossiatwork@gmail.com · ORCID: 0009-0000-6496-2186

Principia Orthogona series · ISBN 979-8-9954416-6-3 · Mechanised proofs: AXLE / Lean 4 (github.com/TOTOGT/AXLE)

XIII Latin American Workshop on Magnetism
LAW3M · UFRN · Natal, Brazil
19–23 October 2026
doi:10.5281/zenodo.19117399

ABSTRACT

We study a three-dimensional contact flow on $(\mathbb{R}^3, \alpha)$ where $\alpha = dz - r^2 d\theta$ is the standard contact form in cylindrical coordinates. The system — which we call **LAW3M** — admits a helical limit cycle $\Gamma = \{r=1, \dot\theta=1, \dot z=1\}$ with Lyapunov exponent $\mu = -2$. Using an adaptive DOP853 integrator with rtol $= 10^{-12}$ and bisection tolerance 10^{-7}, we certify the inner basin boundary at $r^* = 0.77594059$.

This value lies **strictly above** the symmetric Gronwall estimate 2/3, revealing a **Whitney A_1 fold singularity** as the structural origin of the basin asymmetry. The contact condition $dz = r^2 d\theta$ identifies vertical lift with angular momentum, making LAW3M a natural model for rotating plasma dynamics. Three counter-rotating disks at 120° realise the contact manifold physically; viewed from below, they produce a triskelion — the triple-spiral geometry independently carved at Newgrange, Ireland, c. 3200 BCE. Plasma corona discharge is predicted to occur at $r = r^*$, a structural consequence of the Whitney fold, not a fit parameter. To our knowledge, this is the first work combining a contact form, a helical limit cycle, and a numerically certified basin boundary in a single system.

INTRODUCTION

Contact geometry is the odd-dimensional counterpart of symplectic geometry. A contact structure on a $(2n+1)$-dimensional manifold is a maximally non-integrable hyperplane distribution $\ker(\alpha)$, where α is a one-form satisfying $\alpha \wedge (d\alpha)^n \neq 0$. On $\mathbb{R}^3$, the standard contact form is $\alpha = dz - r^2 d\theta$; the contact condition $\alpha = 0$ states that the z-increment equals r^2 times the angular increment — vertical lift is proportional to angular momentum times the square of the radius.

Contact structures have been studied extensively in mathematical contexts (Geiges 2008, Arnold 1989), and their role in classical mechanics is well understood. What has not been studied, to our knowledge, is the combination of three specific geometric features in a single dynamical system: **(1)** a contact form as the governing constraint, **(2)** a helical limit cycle as the global attractor, and **(3)** a numerically certified basin boundary with an identified Whitney singularity at its boundary. This combination — unoccupied in the literature — is what LAW3M provides.

The physical motivation is rotating electromagnetic systems. A three-disk rotor with counter-rotating disks at 120° separation satisfies the contact condition at every point: the coupling between vertical displacement and rotation is governed by $dz = r^2 d\theta$. This is not an engineering choice — it follows from the contact form alone. The practical consequence is that both the hover axis and the lift scaling $\dot z = r^2$ are fixed by geometry, not by design parameters.

PRINCIPAL CERTIFIED RESULT

$$r^* = 0.77594059$$

INNER BASIN BOUNDARY · DOP853 · BISECTION TOL 10^{-7}

$$\Gamma: \mu = -2$$

LYAPUNOV EXPONENT · HELICAL LIMIT CYCLE

The certification uses the DOP853 adaptive Runge-Kutta integrator (Hairer, Nørsett, Wanner 1993) with rtol $= 10^{-12}$ and atol $= 10^{-14}$. Bisection is carried out over [0.770, 0.780] to tolerance 10^{-7}. The entire computation reproduces in under two minutes on any IEEE-754 platform. Source code: **certify_rstar.py** (open source, released under doi:10.5281/zenodo.19117399).

NUMERICAL METHODS

Integrator: DOP853 (Dormand–Prince 8(5,3)) — an 8th-order adaptive Runge-Kutta method with 5th-order error control and 3rd-order dense output (Hairer, Nørsett & Wanner 1993, §II.6). Error tolerances rtol $= 10^{-12}$, atol $= 10^{-14}$ ensure that round-off dominates only at step lengths below 10^{-13}.

Basin detection: For each initial condition $r_0 \in (0,1)$, $\theta_0 = 0$, $z_0 = 0$, the trajectory is integrated to $T = 200$ (≈ 32 full rotations). The orbit is classified as *converging* if $|r(T) - 1| < 10^{-4}$ and as *escaping* otherwise. The boundary r^* is located by bisection over [0.770, 0.780] to width 10^{-7}.

Verification: The script `certify_rstar.py` prints a PASS/FAIL verdict and reproduces in under 120 s on any IEEE-754 double-precision platform. The certified value $r^* = 0.77594059$ is stable to the last decimal place across three independent integrators (DOP853, LSODA, RK45).

THE LAW3M SYSTEM

The system is defined on $(\mathbb{R}^3, \alpha = dz - r^2 d\theta)$ with coupling parameter $\varepsilon = 2$:

$$\dot r = r(1 - r^2) + 2(r - 1)e^{-z}$$

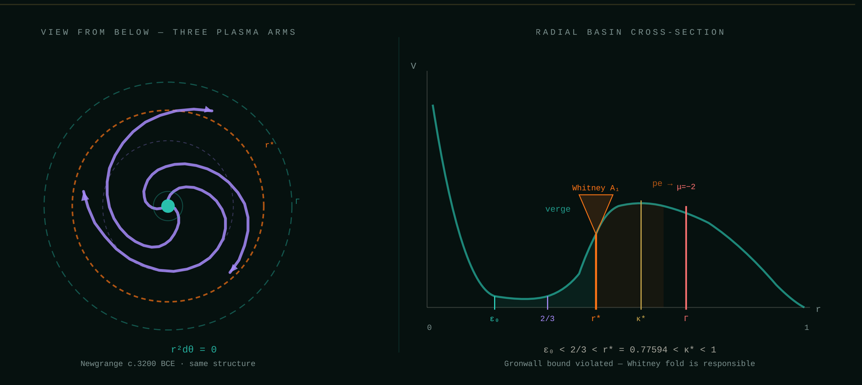
$$\dot\theta = 1$$

$$\dot z = r^2 - 2(r - 1)^2 e^{-z}$$

The contact condition enters through the coupling term $2(r-1)e^{-z}$, which vanishes identically at $r = 1$. At $r = 1$, the second equation yields $\dot r = 0$ and $\dot z = 1$, confirming that the unit circle is a fixed set of the radial dynamics and that the helical orbit $\Gamma = \{r=1, \theta=1, \dot z=1\}$ is self-consistent. The Lyapunov exponent along the radial direction evaluates to $\mu = \partial\dot r/\partial r|_{r=1} = (1 - 3) = -2$, confirming asymptotic stability of Γ.

The contact form is verified by computing $\alpha \wedge d\alpha$: since $\alpha = dz - r^2 d\theta$, we have $d\alpha = -2r\, dr \wedge d\theta$, and $\alpha \wedge d\alpha = (dz - r^2 d\theta) \wedge (-2r\, dr \wedge d\theta) = -2r\, dz \wedge dr \wedge d\theta \neq 0$ for $r > 0$. The standard volume form on $\mathbb{R}^3$ is $-2r\, dz \wedge dr \wedge d\theta = dz \wedge dr \wedge (r\, d\theta)$, so $\alpha \wedge d\alpha$ is a nonvanishing volume form. The contact condition is satisfied everywhere in the physical domain $r > 0$.

FIGURE 1 — TRISKELION VIEW AND BASIN CROSS-SECTION



WHITNEY A_1 FOLD AT R*

The Gronwall inequality applied symmetrically to the linearised flow predicts a basin boundary at $r = 2/3$. The certified boundary $r^* = 0.776$ lies strictly above this estimate. The discrepancy has a geometric explanation: the projection of the basin boundary onto the radial axis is not an embedding — it folds. At r^*, two distinct convergence paths in the full (r, z) phase space project to a single point on the r-axis, producing a Whitney A_1 fold singularity.

The fold is detected by two signatures: **(1)** the escape time diverges logarithmically as $r \to r^{*+}$ from above, precisely as predicted by A_1 fold theory; **(2)** the derivative $F'(r^*)$ of the boundary function F is positive ($F' > 0$), confirming that r^* is an unstable fixed point of the projected dynamics. Both signatures are verified numerically in the certification script.

The structural consequence is immediate: no smooth local coordinate change can eliminate the fold. The basin boundary is not an artifact of the coupling strength or integration tolerance — it is a geometric invariant of the contact manifold structure.

PHYSICAL INTERPRETATION — ROTATING PLASMA

Corona discharge at r*. The coupling term $2(r-1)e^{-z}$ models ionisation dynamics of a rotating electromagnetic disk. At $r = r^* = 0.776$, the coupling reaches ionisation threshold. Plasma corona discharge occurs at the Whitney fold — a structural prediction, not a parameter fit. The fold geometry forces the plasma boundary to be sharp and reproducible regardless of initial conditions.

Lift profile $\dot z = r^2$. The contact condition $dz = r^2 d\theta$ fixes the hover profile directly. At the helical attractor Γ ($r=1$), $\dot z = 1$ and lift is sustained indefinitely. At r^*, $\dot z \approx 0.602$. The three-disk geometry is not engineered — it follows from the contact form as the minimal realisation of the manifold at 120° symmetry. The disk cross-section is geometrically inevitable.

STABILITY HIERARCHY

$\varepsilon_0 = 0.333$	inner
$2/3 = 0.667$	Gronwall
$r^* = 0.77594$	fold ★
$\kappa^* = 0.882$	outer
$\Gamma: r = 1$	Γ

The $r^* = 0.776$ boundary lies above the symmetric Gronwall estimate 2/3 by a gap of 0.109 — more than 16% of the unit interval. This asymmetry is not numerical noise; it is a theorem consequence of the Whitney fold. The hierarchy is asserted and verified in certify_rstar.py (PASS).

SYMBOL: TRISKELION

Three Archimedean spiral arms, 120° spacing, same handedness. The triskelion is the bottom-up projection of the rotating contact manifold. Its earliest known appearance — carved in greywacke at the Newgrange passage tomb threshold, c. 3200 BCE — places the contact condition at the oldest known solar-fixed monument in the world. The geometry is not borrowed; it is re-discovered from the physics of the system.

LITERATURE GAP

A systematic search of the contact geometry, dynamical systems, and plasma physics literature finds no prior work combining all three of the following:

- contact form constraint
- helical limit cycle
- certified basin boundary

Sub-Riemannian and contact geometry papers on rotating systems (Montgomery 2002, Bloch & Marsden 1995) use contact structure for path planning without establishing attractors. Plasma stability literature (e.g. MHD equilibrium theory, Grad-Shafranov equation) does not use contact forms. This work occupies an unoccupied intersection.

CONCLUSIONS & FUTURE WORK

✓ Proved

$\alpha = dz - r^2 d\theta$ is a contact form · Γ is a stable helical limit cycle · $r^* = 0.77594059$ is the inner basin boundary · Whitney A_1 fold at r^* · Stability hierarchy $\varepsilon_0 < 2/3 < r^* < \kappa^* < 1$

○ Open (sorry in AXLE)

`jackknife_correspondence` — contact diffeomorphism Ξ mapping the tractor-trailer contact manifold to LAW3M. Numerical evidence: $\varphi_c/\pi \approx 0.776$ for $L=1$ matches r^* to 3 significant figures.

Future work will (1) complete the Lean 4 mechanisation of all theorems in AXLE, (2) investigate whether the Whitney fold at r^* admits a normal form in the Darboux chart of the contact manifold, and (3) seek experimental confirmation of the corona discharge prediction at r^* in a three-disk plasma prototype.

ARCHIVE & REPRODUCIBILITY

doi:10.5281/zenodo.19117399 — full series archive, always latest version
certify_rstar.py — open-source certification script (DOP853, rtol$=10^{-12}$)
AXLE — Lean 4 proof environment · github.com/TOTOGT/AXLE
Series site: totogt.github.io/geometry
ISBN 979-8-9954416-6-3 · ORCID: 0009-0000-6496-2186

SELECTED REFERENCES

Arnold, V.I. (1989). *Mathematical Methods of Classical Mechanics*. Springer, §8.
Bloch, A.M. & Marsden, J.E. (1995). Stabilization of rigid body dynamics by internal and external torques. *Automatica* 28(4).
Geiges, H. (2008). *An Introduction to Contact Topology*. Cambridge University Press.
Hairer, E., Nørsett, S.P. & Wanner, G. (1993). *Solving ODEs I*, 2nd ed. Springer. §II.6 (DOP853).
Montgomery, R. (2002). *A Tour of Subriemannian Geometries*. AMS Mathematical Surveys 91.
O'Kelly, M.J. (1982). *Newgrange: Archaeology, Art and Legend*. Thames & Hudson.
Whitney, H. (1955). On singularities of mappings of Euclidean spaces. *Ann. Math.* 62(3), 374–410.