Helical Attractors on Contact 3-Manifolds

Atratores Helicoidais em Variedades de Contato

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The system

On the contact 3-manifold $(\mathbb{R}^3, \alpha)$ with $\alpha = dz - r^2\, d\theta$ and Reeb field $R = \partial_z$, write a state as (r, θ, z). The system dm^3 is

$$\dot r = r(1 - r^2) + \varepsilon(r-1)e^{-z}$$
$$\dot\theta = 1$$
$$\dot z = r^2 - \varepsilon(r-1)^2 e^{-z}$$

with $\varepsilon = 2$. The cylinder $\Gamma = \{r = 1\}$ is invariant and carries the helical flow $\dot\theta = \dot z = 1$ — a lift of the Reeb orbit.

The question: does Γ attract?

Main theorem

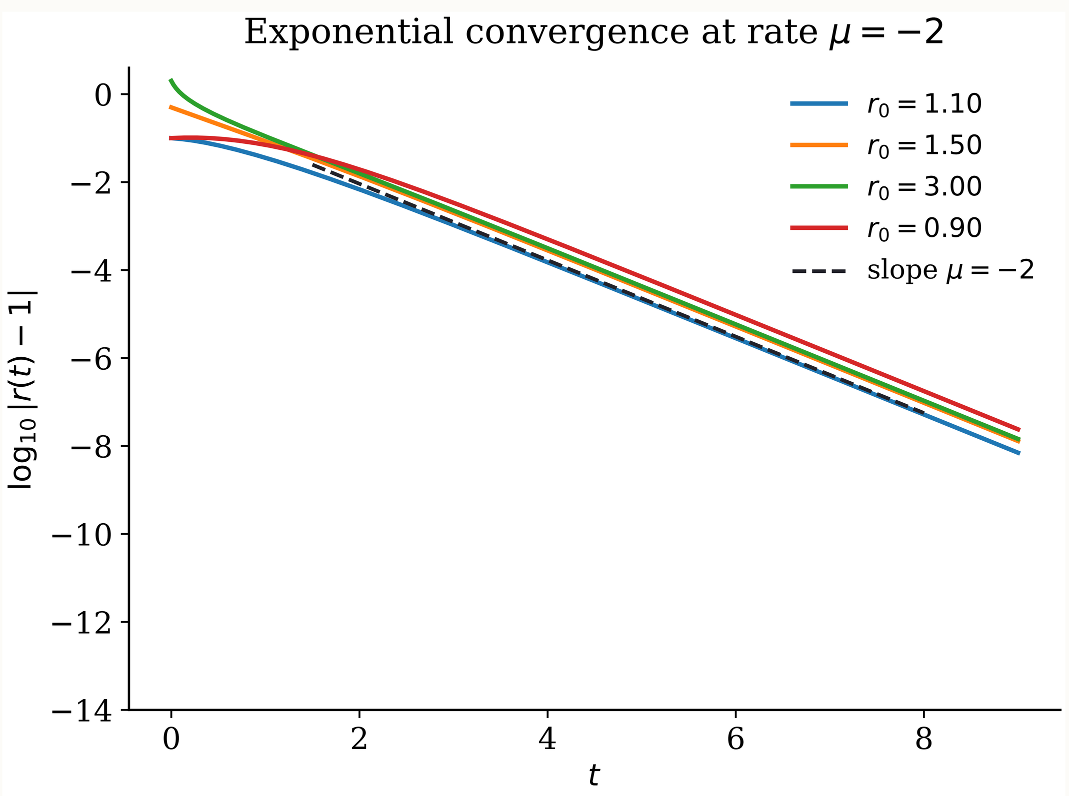
THEOREM 2.1

For initial data with $|r(0) - 1| < 1/3$ and $z(0) \geq \log 2$, the solution exists for all $t \geq 0$ and

$$|r(t) - 1| \;\leq\; C\,|r(0) - 1|\, e^{-2t},$$

so every such orbit converges exponentially to Γ at the rate $\mu = -2$ — the Jacobian eigenvalue of the radial part at $r = 1$.

Proof. Gronwall. Squaring the radial equation gives $\frac{1}{2}\frac{d}{dt}u^2 \leq -2u^2 - 3u^3 - u^4 + \varepsilon u^2 e^{-z}$ with $u = r - 1$; the hypothesis on $z(0)$ forces the last term to be dominated on the relevant window. ☐



Basin asymmetry

Numerically the basin is *not* symmetric about Γ. Outward, every tested initial condition converges. Inward, orbits collapse — $r \to 0$, $z \to -\infty$ in finite time.

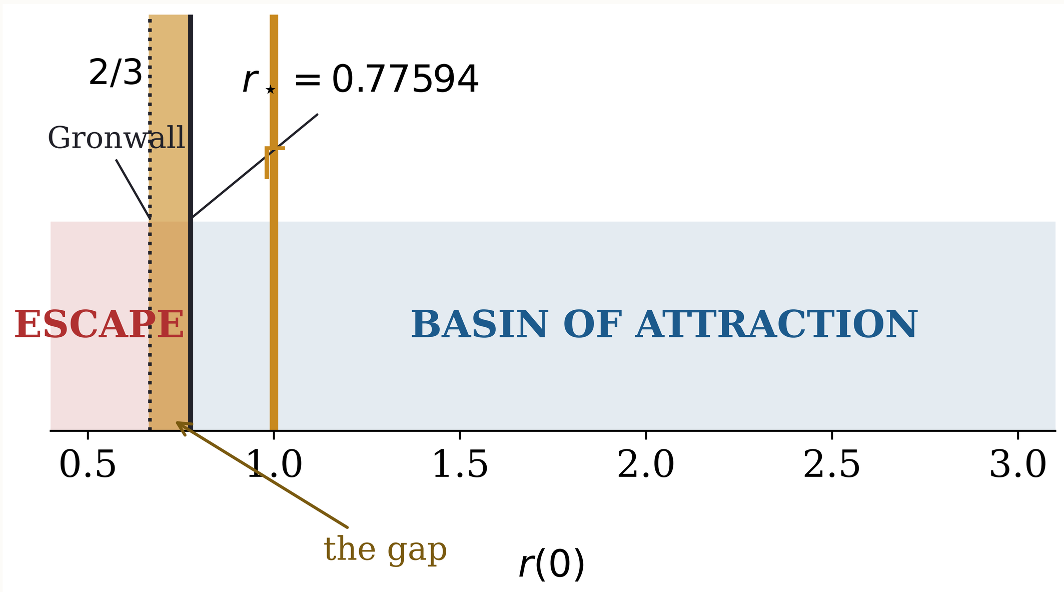
CERTIFIED INNER BOUNDARY

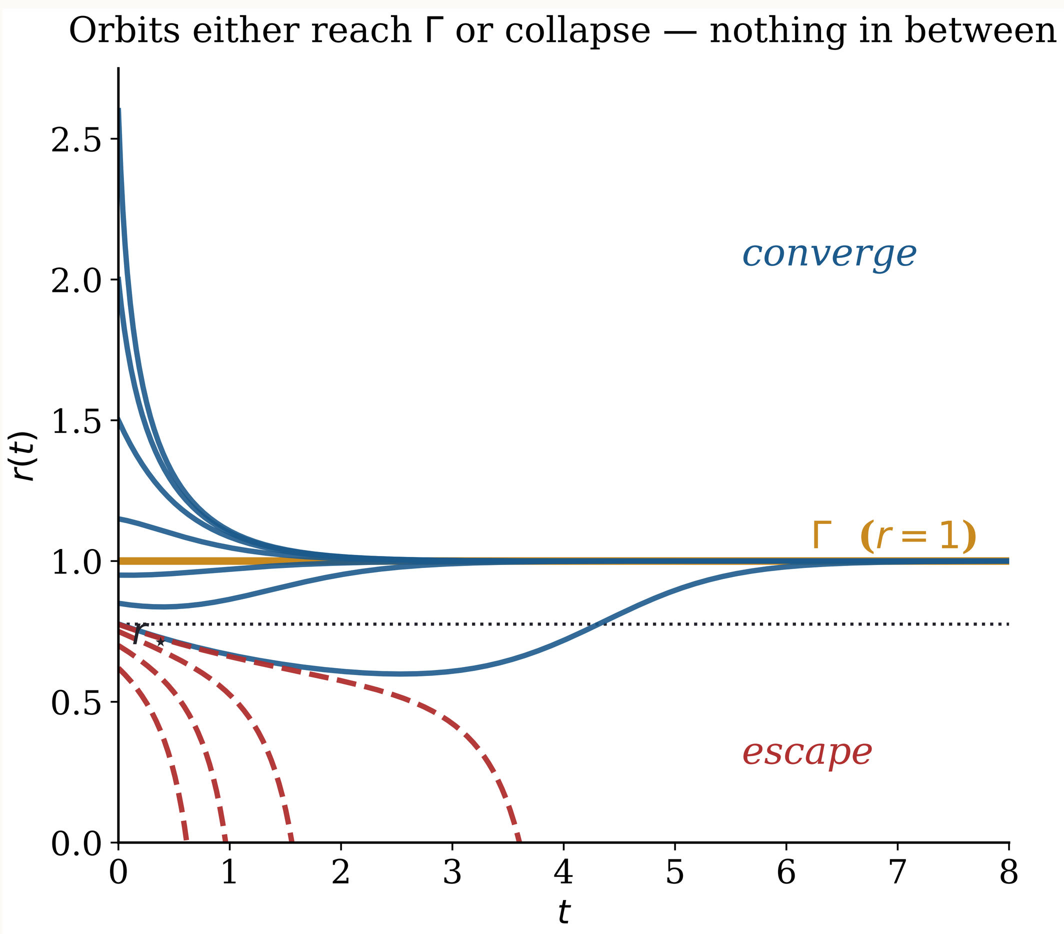
$$r_\star = 0.77594059$$

DOP853 · rtol 10^{-12} · bisection to 10^{-7}

reproducible on any IEEE-754 platform

Since $r_\star > 2/3$, the interval $(2/3,\, r_\star)$ holds initial data *inside* the Gronwall ball that nevertheless escape. This does not contradict Theorem 2.1 — the second hypothesis $z(0) \geq \log 2$ is exactly what excludes them.





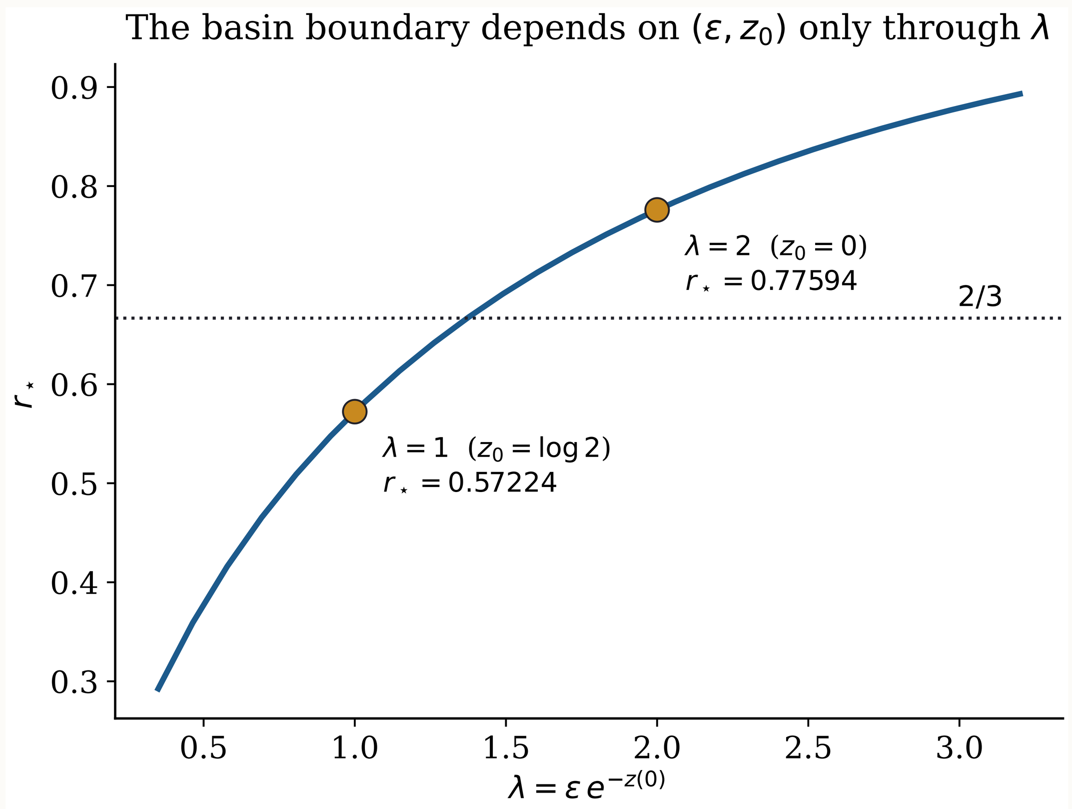
A scaling law

The pair $(\varepsilon, z(0))$ enters the dynamics *only* through

$$\lambda \;=\; \varepsilon\, e^{-z(0)},$$

since ε and e^{-z} never occur apart. Every basin quantity is a function of λ alone — verified to 10^{-12} across independent (ε, z_0) pairs.

λ	$r_\star$	
0.5	0.377975	
1.0	**0.572235**	$(z_0 = \log 2)$
2.0	**0.775941**	$(z_0 = 0)$
3.0	0.878915	



At $\lambda = 1$ one finds $r_\star < 2/3$: the Gronwall ball sits strictly inside the basin. **The hypothesis $z(0) \geq \log 2$ is necessary, not cosmetic.**

Contact geometry

The asymmetry $r_\star \neq 2 - r_\star$ is diagnostic. It is the fingerprint of the term $\varepsilon(r-1)^2 e^{-z}$ in $\dot z$, which is not symmetric under $u \mapsto -u$ because it couples to the *non-integrable* direction of α. A symmetric basin would mean the attractor reduces to a 2D problem transverse to R; the observed asymmetry says the full 3-manifold contact structure is essential.

Status of claims

Claim	Status
Γ invariant, $\mu = -2$	proved
Scaling law in λ	proved
$r_\star = 0.77594059$	certified numerically
Closed form for $r_\star$	**open**
Lipschitz bound (AXLE #12)	**open**

Reproduce: `certify_rstar.py` — DOP853, deterministic bisection, MIT.

Livro vivo (B3 · Vol. III): `totogt.github.io/geometry`

Capítulo 10 (Vol. IV): `totogt.github.io/GTCT`

Formalização: `github.com/TOTOGT/AXLE`

What is new here

1. The scaling law

(ε, z_0) collapse to the single parameter $\lambda = \varepsilon e^{-z_0}$. One curve $r_\star(\lambda)$ replaces a two-parameter family.

2. The hypothesis earns its place

$z(0) \geq \log 2$ is what pulls $r_\star$ below $2/3$. Drop it and the Gronwall ball provably contains escaping orbits.

3. A number you can check

$r_\star$ to eight figures, by deterministic bisection, from a script that runs in seconds on any machine.

Submissões companheiras: **EXP13** Cimática com Máquinas (Saguão do CETECH, todos os dias) • **MC48** Minicurso (terça 04/08, E1LAB) • **CO144** O Princípio do Cajueiro (quarta 05/08, F6) • **OF53** A Escada de Recorrência (sexta 07/08, H3)