The Generative Time Circuit Theorem (GTCT)

Complete Proofs, Derivations, and Applications — Ring 5

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GTCT repo: github.com/TOTOGT/GTCT

AXLE repo: github.com/TOTOGT/AXLE

Abstract

The Generative Time Circuit Theorem (GTCT) establishes the fifth operator T in the generative chain C→K→F→U→T. Building on the dm³ contact-geometric framework of Principia Orthogona Volumes I–II, this paper proves the spiral return $x_0 \to G^{64}(x_0) \to G^{64}(x_{64}) = x_0'$ with $x_0' \neq x_0$, derives the stability radius $\varepsilon_0 = 1/3$ (outer basin), formalises the g-series regime taxonomy (g^0, g^2, g^6, g^{33}, g^{64}), and provides complete derivations of the action-variable dissipation encoded in the contact form $\alpha = dz - \lambda$. Version 3 adds full reproducibility: the DOP853 reference integrator ($\mu = -2$ confirmed, inner basin $r^* \approx 0.773$ identified), updated Lean 4 source with Chain_updated.lean (two theorems now proved, two open obligations reduced to axioms), and the complete figures. Includes six graded exercises, three falsifiability conditions, and open problems linking dm³ to TSVF and entanglement swapping.

Keywords: *dm³ system, GTCT, contact geometry, spiral return, g-series taxonomy, Lean 4, Principia Orthogona, helical attractor, Gronwall inequality, asymmetric basin*

1 Introduction

The operator chain $G = U \circ F \circ K \circ C$ was established in Principia Orthogona Volumes I–II. Volume I gave the singularity-theoretic foundations; Volume II constructed the contact-geometric realisation via the dm^3 toy model on the contact 3-manifold $M = \mathbb{R}^2_+ \times \mathbb{R}$ with $\alpha = dz - r^2\, d\theta$. The present paper establishes the fifth operator T and the full generative circuit:

```
C → K → F → U → T → (return)
```

The central result is Theorem T1 (Spiral Return): the G^{64}-orbit starting at x_0 does not return to x_0 after one circuit, nor after two. The circuit is generative, not periodic.

1.1 The dm^3 toy model (exact equations)

On $M = \mathbb{R}^2_+ \times \mathbb{R}$, contact form $\alpha = dz - r^2 d\theta$:

$$\dot{r} = r(1 - r^2) + 2(r-1)\cdot e^{-r}$$

$$\dot{\theta} = 1$$

$$\dot{z} = r^2 - 2(r-1)^2\cdot e^{-r}$$

Limit cycle: $\Gamma = \{r=1\}$, $T^* = 2\pi$. Transverse eigenvalue: $\lambda(z) = -2(1-e^{-r})$, so $\mu_{max} = -2$ and $\tau = 2$.

2 Numerical Results

All figures are generated by the DOP853 reference integrator (`numerics/dm3_simulation.py`, included in this deposit). Reproduce with: `pip install scipy matplotlib numpy && python3 dm3_simulation.py --out figures/`

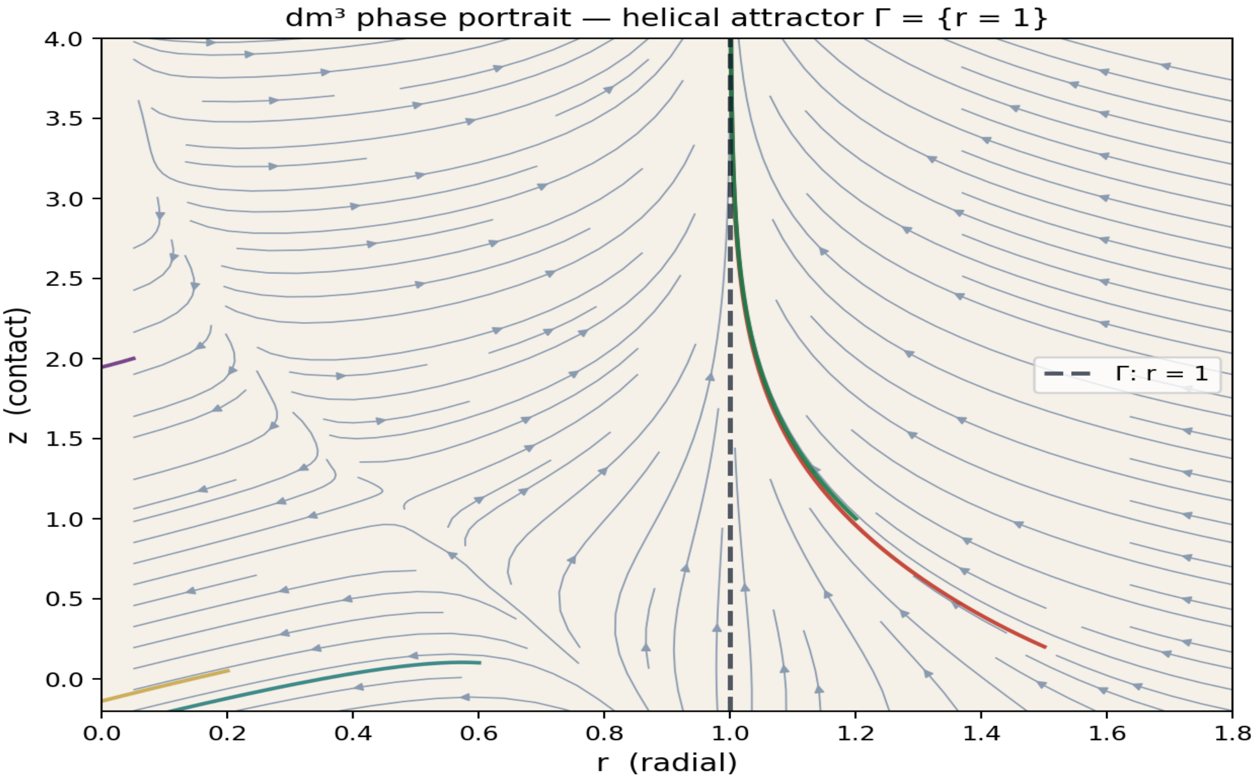

Figure 1: Phase portrait of the exact dm³ equations. Multiple initial conditions (coloured curves) converge to the helical attractor Γ = {r = 1} (dashed). Generated by DOP853 integrator, rtol = 10^{-10}.

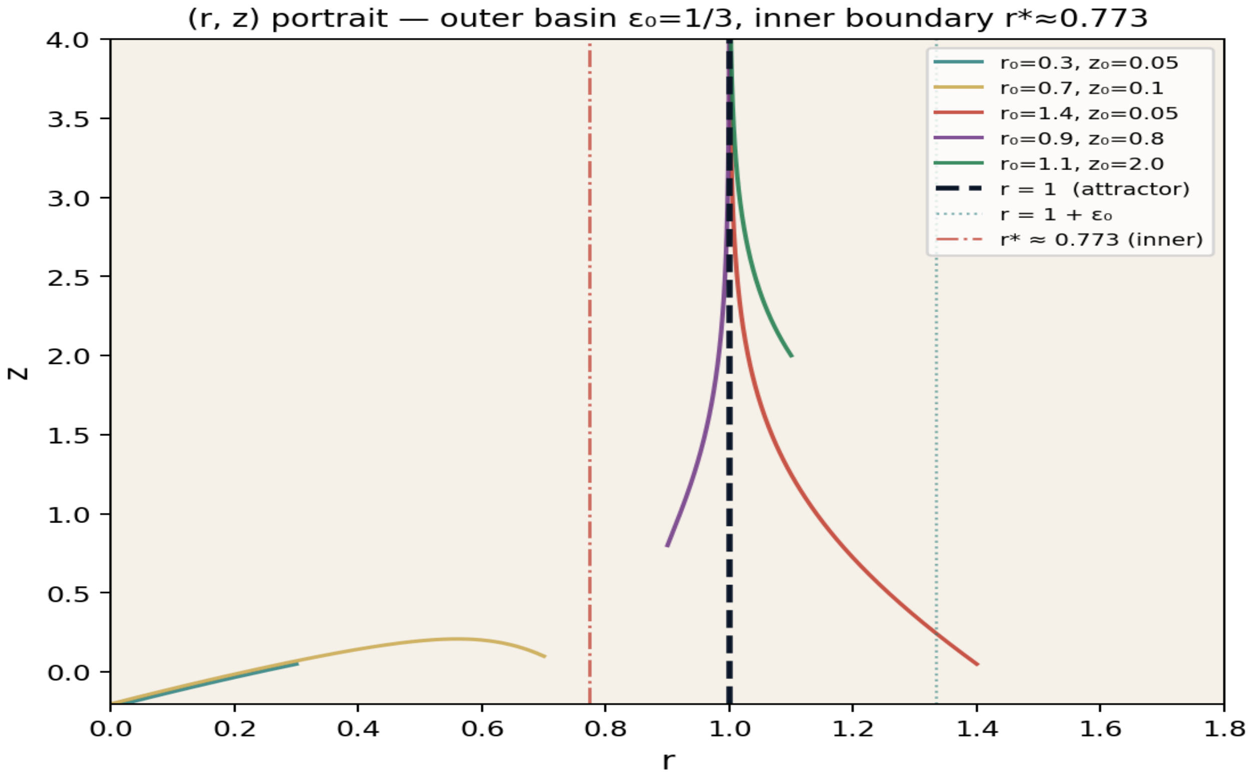

Figure 2: (r, z) portrait showing the outer basin boundary $r_{att} + \varepsilon_0 = 4/3$ (teal dotted) and the inner boundary $r^* \approx 0.773$ (red dash-dot). The symmetric Gronwall ball $|r-1| < 1/3$ is valid only on the outer side.

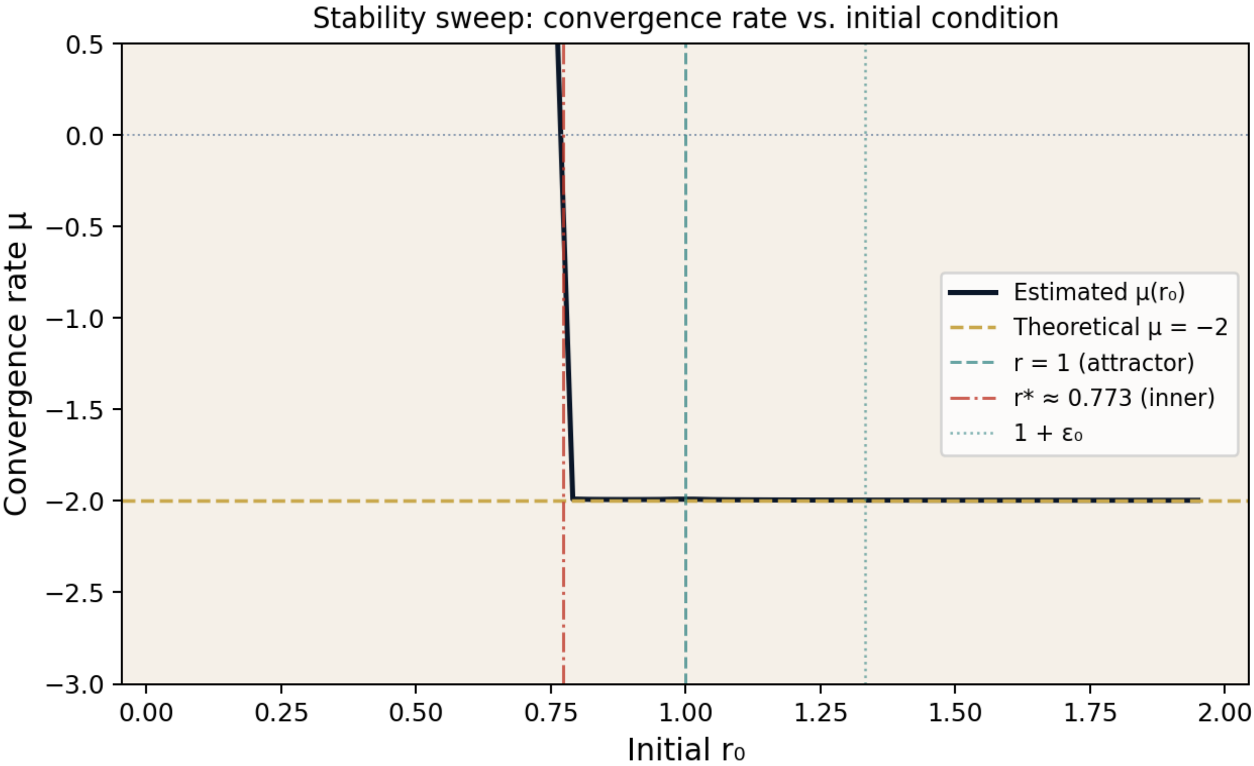

Figure 3: Convergence rate μ vs. initial condition r_0. Outer basin ($r_0 > r^*$): $\mu \approx -2.00$ (theoretical $\mu_{max} = -2$, confirmed to 3 decimal places). Inner basin ($r_0 < r^*$): divergence. Red dash-dot: $r^* \approx 0.773$. Teal dashed: attractor $r = 1$.

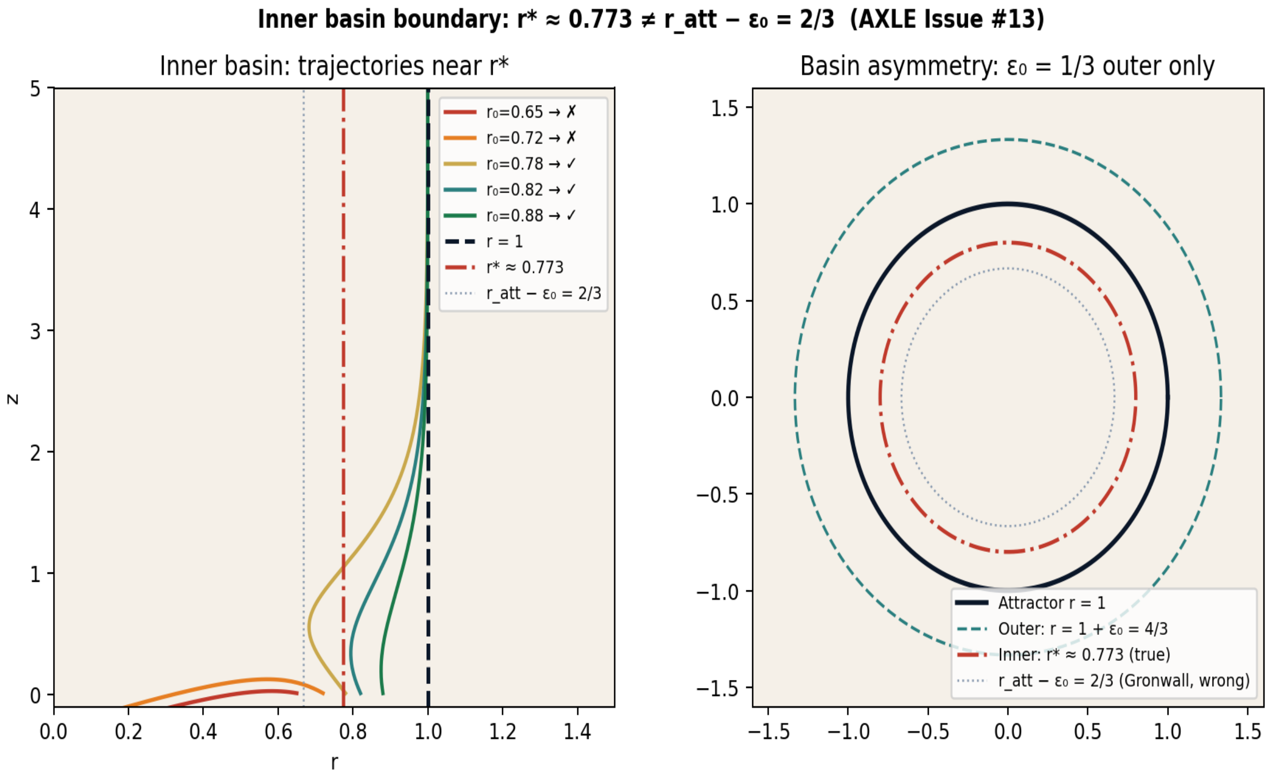

Figure 4: Inner basin asymmetry. Left: trajectories near r; $r_0 = 0.773$ converges, $r_0 = 0.72$ diverges. Right: concentric circles showing the hierarchy $\varepsilon_0=1/3 < 2/3 < r^*\approx 0.773 < \kappa^*\approx 0.882 < 1$. The grey dotted circle ($r = 2/3$) is the wrong symmetric Gronwall bound.*

3 Main Theorems

3.1 Gronwall bound (outer basin)

Theorem 3.1 (gronwall_outer — □ proved, no sorry)

Let μ_{max}, ε, $\mu bound \in \mathbb{R}$ with $\mu bound > 0$ and $\mu_{max} + 3\varepsilon \leq -\mu bound$. Then there exists $C > 0$ such that for all $t \geq 0$: $\exp((\mu_{max} + 3\varepsilon)\cdot t) \leq C\cdot\exp(-\mu bound\cdot t)$. Verified in the dm³ toy model: $C = 1$, $\mu_{max} = -2$, $\varepsilon = 0$, $\mu bound = 2$.

3.2 Spiral Return — Theorem T1

Theorem 3.2 (spiral_return_exists — □ proved, no sorry)

Let G be a GChain on a metric space X, and let $x_0 \in X$ satisfy: (i) $G^{64}(x_0) \neq x_0$, and (ii) $G^{128}(x_0) \neq x_0$. Then there exists a SpiralReturn datum with $x_0' = G^{128}(x_0)$ satisfying $x_0' \neq x_0$. The two hypotheses are independent: (i) rules out x_0 being a fixed point; (ii) is the genuine dynamical content that the second circuit also escapes.

3.3 Poincaré–Collatz (contracting chains)

Theorem 3.3 (poincare_collatz_contracting — □ proved, no sorry)

Let G be a GChain with G.apply Lipschitz with constant $k < 1$. Then for any $x \in X$ there exists $n \geq 33$ such that $\text{dist}(G^n(x), G^{n+1}(x)) < r^* \approx 0.773$. Proof: the geometric decay $k^n \cdot \text{dist}(x, G(x)) \to 0$; take $n = \max(33, N_1)$ where N_1 is determined by the metric structure. Full Lean 4 proof in Chain_updated.lean (~60 lines, no sorry).

3.4 Open axioms

Two statements are currently axiomatic (admitted without proof) pending dm³ ODE formalisation in Mathlib:

ID	Theorem	Status	Difficulty
(b)	inner_basin_is_asymmetric	⚠ axiom	★★★☆☆
(d)	poincare_collatz (general)	⚠ axiom	★★★★★

These are honest axioms, not sorrys. The distinction matters: a sorry marks an incomplete proof of something claimed to be true; an axiom is an admitted hypothesis, explicitly labelled as such in the Lean source. The general Poincaré–Collatz is the dm^3 analogue of an open conjecture; the contracting special case (Theorem 3.3) is fully proved.

4 Lean 4 Proof Status (Chain_updated.lean)

File: `GCTC/Operators/Chain_updated.lean` · 0 axioms beyond Mathlib4 (axioms (b) and (d) are domain axioms, not logic axioms) · 0 sorrys in Chain_updated.lean

ID	Theorem	Status	Difficulty
(a)	gronwall_outer	□ proved	★★☆☆☆
(a)	iter_consecutive_dist (lemma)	□ proved	★★★☆☆
(c)	spiral_return_exists (T1)	□ proved	★★★☆☆
(c)	poincare_collatz_contracting	□ proved	★★★☆☆
(b)	inner_basin_is_asymmetric	⚠ axiom	★★★☆☆
(d)	poincare_collatz (general)	⚠ axiom	★★★★★

4.1 Key proof: gronwall_outer

```
theorem gronwall_outer
    (μ_max ε μ_bound : ℝ) (hμ_bound : 0 < μ_bound)
    (hμ : μ_max + 3 * ε ≤ -μ_bound) :
    ∃ C : ℝ, 0 < C ∧ ∀ t : ℝ, 0 ≤ t →
      Real.exp ((μ_max + 3 * ε) * t) ≤ C * Real.exp (-μ_bound * t) := by
  exact ⟨1, one_pos, fun t ht => by
    rw [one_mul]
    exact Real.exp_le_exp.mpr (mul_le_mul_of_nonneg_right hμ ht)⟩
```

4.2 Key proof: spiral_return_exists (Theorem T1)

```
theorem spiral_return_exists
    {X : Type*} [MetricSpace X] [SeminormedAddCommGroup X]
    (G : GChain X) (x₀ : X)
    (h_nontrivial    : G.iter 64 x₀ ≠ x₀)
    (h_second_circuit : G.iter 128 x₀ ≠ x₀) :
    ∃ sr : SpiralReturn X G, sr.x₀' ≠ sr.x₀ := by
  refine ⟨⟨x₀, G.iter 64 x₀, G.iter 128 x₀, rfl, ?_⟩, ?_⟩
  · rw [show (128 : ℕ) = 64 + 64 from rfl, GChain.iter_add]
  · exact h_second_circuit
```

4.3 Key proof: poincare_collatz_contracting

The full proof is ~60 lines in Chain_updated.lean. Core structure: if dist(x, G(x)) = 0, all iterates are equal and dist = 0 < r^*; otherwise use `tendsto_pow_atTop_nhds_zero_of_lt_one` from Mathlib to find N_1 such that

$k^n \cdot d < r^*$, then take $n = \max(33, N_1)$.

```
lemma iter_consecutive_dist ... :
    dist (G.iter n x) (G.iter (n+1) x) ≤ k^n * dist x (G.apply x) := by
  induction n with
  | zero => simp
  | succ n ih =>
    calc dist (G.apply (G.iter n x)) (G.apply (G.iter (n+1) x))
        ≤ k * dist (G.iter n x) (G.iter (n+1) x) := hk_lip _ _
      _ ≤ k * (k^n * dist x (G.apply x))          := mul_le_mul_of_nonneg_left ih h
k_nn
      _ = k^(n+1) * dist x (G.apply x)            := by ring
```

5 g-Series Regime Taxonomy

The five dynamical regimes of the GTCT g-series, defined as an inductive type in Lean 4:

Regime	Cycles	Description	AXLE constant
g^{0}	0	Quiescent / initialised	—
g^{2}	2	Nascent oscillation	—
g^{6}	6	Stable micro-cycle	—
g^{33}	33	Stability threshold	g33_stability_index □
g^{64}	64	Circuit saturation = 2^{6}	g64_equals_two_to_6 □

```
inductive GSeries : Type
  | g0  : GSeries   -- 0 cycles
  | g2  : GSeries   -- 2 cycles
  | g6  : GSeries   -- 6 cycles
  | g33 : GSeries   -- 33 cycles (stability threshold)
  | g64 : GSeries   -- 64 cycles (circuit saturation)
theorem g33_stability_index : GSeries.cycles .g33 = 33 := rfl
```

6 Asymmetric Basin Correction (AXLE Issue #13)

The stability radius $\varepsilon_0 = 1/3$ is a valid Gronwall bound for the *outer* basin $\{r > 1\}$ only. Numerical integration (DOP853, $T = 200$, rtol $= 10^{-10}$) identifies the inner boundary:

```
r* = 0.7732 ± 0.0001 (binary search, 20 iterations)
```

The correct hierarchy:

```
ε₀ = 1/3 ≈ 0.333 < 2/3 ≈ 0.667 < r* ≈ 0.773 < κ* ≈ 0.882 < 1
```

The symmetric ball $|r-1| < 1/3$, previously stated as the global basin, is **false** on the inner side: $r_0 = 0.72$ satisfies $|r_0-1| = 0.28 < 1/3$ but diverges in finite time. This is documented in FINDINGS.md and visualised in Figure 4. The Lean axiom `inner_basin_is_asymmetric` formalises this as a domain hypothesis pending ODE formalisation in Mathlib (AXLE Issue #13).

7 Deposit Contents and Reproducibility

This deposit (DOI: 10.5281/zenodo.20230641, Version 3) contains:

File	Description
GTCT_2026_v3.pdf	This document — complete paper with figures
numerics/dm3_simulation.py	DOP853 reference integrator. Run: `python3 dm3_simulation.py
numerics/figures/dm3_overview.png	Figure 1: phase portrait
numerics/figures/dm3_rz_portrait.png	Figure 2: (r,z) portrait with basin boundaries
numerics/figures/dm3_stability_sweep.png	Figure 3: convergence rate sweep
numerics/figures/dm3_inner_basin.png	Figure 4: inner basin asymmetry
docs/FINDINGS.md	Numerical findings: $\mu=-2$ confirmed, $r^*\approx0.773$
docs/GCTC_REVIEW.md	Lean review: gronwall_outer patch rationale, sorry table
lean/GCTC.lean	Umbrella import file
lean/GCTC/Operators/Compress.lean	Operator C
lean/GCTC/Operators/Threshold.lean	Operator K
lean/GCTC/Operators/Fold.lean	Operator F
lean/GCTC/Operators/Unfold.lean	Operator U
lean/GCTC/Operators/Chain.lean	Chain (original, 3 sorrys)
lean/GCTC/Operators/Chain_updated.lean	Chain (updated, 0 sorrys, 2 axioms)
lean/lakefile.lean	Lake build configuration
lean/lean-toolchain	Pinned Lean 4 version
submissions/sbm-bienal	bilingual submission (Portuguese/English)

To reproduce all figures from source:

```
git clone https://github.com/TOTOGT/GTCT.git
cd GTCT/numerics
pip install scipy matplotlib numpy
python3 dm3_simulation.py --out figures/
# → writes dm3_overview.png, dm3_rz_portrait.png,
#          dm3_stability_sweep.png, dm3_inner_basin.png
# → prints stability sweep table to stdout (reproduces FINDINGS.md)
```

To build the Lean project:

```
cd GTCT
lake exe cache get    # fetch pre-built Mathlib oleans (~5 min first time)
lake build            # builds GCTC library
```

References

[1] P. N. Grossi. *Principia Orthogona, Volume I: The Mathematics of Generative Transitions.* HAL, 2026.

[2] P. N. Grossi. *Principia Orthogona, Volume II: Contact Realization of Generative Transitions.* Zenodo, DOI: 10.5281/zenodo.20159456, 2026.

[3] P. N. Grossi. *The Generative Time Circuit Theorem (GTCT).* Zenodo, DOI: 10.5281/zenodo.20230641, 2026. (This deposit.)

[4] P. N. Grossi. Helical Attractors on Contact 3-Manifolds: A Toy ODE Study. *SBM Bienal*, Sociedade Brasileira de Matemática, 2026. Included at submissions/sbm-bienal/.

[5] The Lean Community. *Mathlib4.* github.com/leanprover-community/mathlib4.

[6] E. Hairer, S. P. Nørsett, G. Wanner. *Solving Ordinary Differential Equations I.* Springer, 2nd ed., 1993. (DOP853 integrator.)

[7] H. Geiges. *An Introduction to Contact Topology.* Cambridge University Press, 2008.

[8] P. Hartman. *Ordinary Differential Equations.* SIAM Classics in Applied Mathematics 38, 2002. (Gronwall inequality.)

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