A Contact–Geometric Toy Model on the Solid Cylinder: Transverse Stability, Closure–Dependent Escape, Degenerate Hopf Structure, and a Cosmological No–Go

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Abstract

We study the transverse dynamics near the helical periodic orbit $\Gamma = \{r = 1\}$ of a dissipative flow on the contact 3–manifold $(\mathbb{R}_{>0} \times \mathbb{S}^1 \times \mathbb{R},\ \alpha = \mathrm{d}z - r^2\,\mathrm{d}\theta)$, in which the transverse relaxation is modulated by a factor $1 - e^{-z}$. Six results are established, each verified both numerically (double–precision `DOP853`, `rtol`$= 10^{-10}$) and, where tractable, formally (Lean 4 / Mathlib4). (i) The linearised transverse equation admits a closed form, and its Lyapunov exponent $\mu = -2$ is *preserved* because the e^{-z} modulation is integrable along the z–flow, contributing only a bounded additive distortion. (ii) The instantaneous contraction rate changes sign on the *neutral line* $z = 0$: relaxation for $z > 0$, expansion for $z < 0$, with the sub–line flow the exact time reversal of the super–line flow. (iii) The *global* fate is closure–dependent: a cubic closure exhibits genuine finite–time escape below an analytically characterised basin boundary $z_0^*(\varepsilon_0)$, whereas any bounded closure converges globally — so "where it will not converge" is a real prediction only for super–linear closures. (iv) At the axis $r = 0$ the flow carries a *degenerate* Hopf bifurcation whose first Lyapunov coefficient vanishes at onset, pinning the cycle radius at 1 rather than opening it as $\sqrt{\lambda}$. (v) A kinematic reading $z = \ln a$ identifies $\dot{z} = 1$ with de Sitter expansion and e^{-z} with the inverse scale factor; we state precisely the three obstructions that prevent this correspondence from being promoted to a dynamical model. (vi) We prove a *no–go*: under any contact–Hamiltonian flow of α with constant rotation, the transverse rate is locked to the derivative of the expansion rate, $c(z) = H'(z)$, whence a transverse attractor and a de Sitter expansion rate cannot coexist. A graded problem set with solutions accompanies the paper.

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1 Setup and the model

Let $M = \{(r,\theta,z) : r > 0,\ \theta \in \mathbb{S}^1,\ z \in \mathbb{R}\}$ carry the contact 1–form

$$\alpha \;=\; \mathrm{d}z - r^2\,\mathrm{d}\theta. \tag{1}$$

One checks $\alpha \wedge \mathrm{d}\alpha = -2r\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}z \neq 0$, so α is contact and the Reeb field is $R = \partial_z$. Writing $p := r^2$ for the momentum conjugate to θ, the form is the standard Hamilton–Jacobi contact form $\alpha = \mathrm{d}z - p\,\mathrm{d}\theta$.

We study the flow

$$\dot r \;=\; f(r)\left(1 - e^{-z}\right), \qquad \dot\theta \;=\; 1, \qquad \dot z \;=\; 1, \tag{2}$$

where the *closure* $f : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies

$$f(1) = 0, \qquad f'(1) = -2, \qquad f(r)\,(r-1) < 0 \ \ (r > 0,\ r \neq 1). \tag{3}$$

Conditions (3) make $\Gamma = \{r = 1\}$ a periodic orbit of period $T^* = 2\pi$ (a helix in (r,θ,z), since $\dot\theta = \dot z = 1$), with clean transverse eigenvalue -2 when the modulation saturates ($z \to +\infty$). Two canonical closures obey (3):

$$f_{\mathrm{cub}}(r) = r - r^3 \quad \text{(super–linear)}, \qquad f_{\mathrm{sat}}(r) = \frac{-2(r-1)}{1+(r-1)^2} \quad \text{(bounded)}. \tag{4}$$

Both have $f(1) = 0$, $f'(1) = -2$; they differ only in their large–r behaviour, and §4 shows that this difference alone decides global dynamics.

Set $\varepsilon := r - 1$. Linearising (2) about Γ using (3) gives the *transverse linear equation*

$$\dot\varepsilon \;=\; f'(1)\,(1 - e^{-z})\,\varepsilon \;=\; 2\,\varepsilon\,(e^{-z} - 1), \qquad z = z_0 + t, \tag{5}$$

which is independent of the closure. All near–helix statements (§§2–3) follow from (5); the closure enters only in the global statements of §4.

2 Closed form and the preserved exponent

Theorem 1 (Closed form; invariance of the exponent)**.** *The solution of* (5) *with* $z = z_0 + t$ *is*

$$\varepsilon(t) \;=\; \varepsilon_0\,\exp\!\Big(-2t + 2e^{-z_0}\left(1 - e^{-t}\right)\Big). \tag{6}$$

Consequently, for $\varepsilon_0 \neq 0$*,*

$$\mu \;:=\; \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\varepsilon(t)|}{|\varepsilon_0|} \;=\; -2. \tag{7}$$

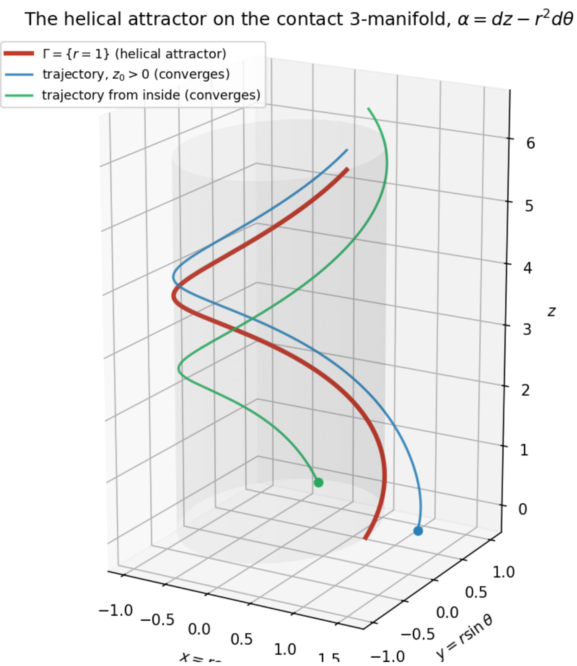

Figure 1: The helical attractor $\Gamma = \{r = 1\}$ (red) on the contact manifold, with two trajectories of (2) (cubic closure) started off the cylinder at $z_0 > 0$ and converging onto it. The transverse coordinate is r; the flow advances uniformly in θ and z.

Proof. Separating variables in (5), $\mathrm{d}(\ln|\varepsilon|) = 2(e^{-(z_0+t)} - 1)\,\mathrm{d}t$, and integrating,

$$\ln|\varepsilon(t)| - \ln|\varepsilon_0| = 2\int_0^t \big(e^{-(z_0+s)} - 1\big)\,\mathrm{d}s = 2\big(-e^{-(z_0+t)} + e^{-z_0}\big) - 2t = -2t + 2e^{-z_0}(1 - e^{-t}),$$

which is (6). The bracketed term is bounded by $2e^{-z_0}$ uniformly in t, so $\frac{1}{t}\ln|\varepsilon/\varepsilon_0| = -2 + \frac{2e^{-z_0}(1-e^{-t})}{t} \to -2$. $\square$

Remark 2 (Why the modulation is invisible to the exponent)**.** Because z advances linearly, $\int_0^\infty e^{-(z_0+s)}\,\mathrm{d}s = e^{-z_0} < \infty$: the e^{-z} term is *integrable along the flow*. It therefore shifts $\ln|\varepsilon|$ by a bounded amount and can never alter the asymptotic rate. The verified constant $\mu = -2$ survives the new ingredient untouched; only the transient is reshaped.

3 The neutral line $z = 0$

Theorem 3 (Sign of the instantaneous rate)**.** *Let* $\rho(z) := 2(e^{-z} - 1)$ *be the instantaneous transverse rate,* $\frac{\mathrm{d}}{\mathrm{d}t}\ln|\varepsilon| = \rho(z)$*. Then*

$$\rho(z) > 0 \ (z < 0), \qquad \rho(0) = 0, \qquad \rho(z) < 0 \ (z > 0).$$

The locus $z = 0$ *is non–hyperbolic (loss of transverse hyperbolicity), and for* $z < 0$ *the transverse flow is the exact time reversal of the* $z > 0$ *flow.*

Proof. ρ has the sign of $e^{-z} - 1$, hence of $-z$; it vanishes only at $z = 0$. Writing $\dot\varepsilon = \lambda(z) f_1(\varepsilon)$ with $\lambda(z) = 1 - e^{-z}$ and $f_1(\varepsilon) = -2\varepsilon + O(\varepsilon^2)$, the factor λ changes sign at $z = 0$, and $\lambda(-z') = -\lambda(z')\,e^{-z'}$

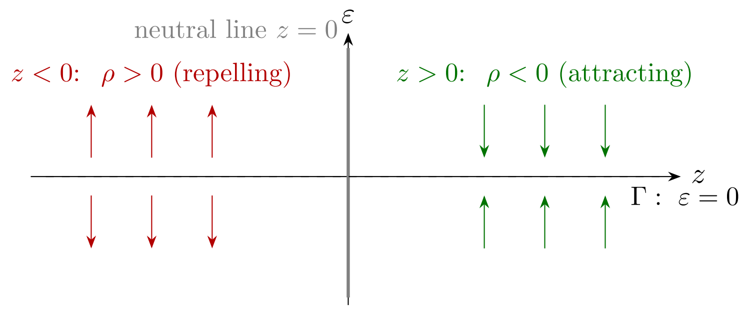

Figure 2: Transverse regime diagram in the (z, ε) plane. The helix $\varepsilon = 0$ is transversally attracting for $z > 0$ and repelling for $z < 0$; $z = 0$ is the non–hyperbolic neutral line. Every trajectory sweeps $z = z_0 + t$ rightward at unit rate, so it undergoes a *slow passage* through this stability reversal.

shows the two half–line flows are related by time reversal up to the positive reparametrisation $e^{-z'}$. □

The two half–lines and the neutral locus are summarised in the regime diagram, Fig. 2.

Remark 4 (Swept (dynamic) bifurcation)**.** Since every orbit carries its own control parameter z through the reversal at unit rate, the transient "grow while $z < 0$, decay while $z > 0$" is the canonical signature of *slow passage through a bifurcation* [3]. This is the correct technical home for the "eye"–like transient concentration near Γ: not a static bifurcation of a phase portrait, but a swept passage through a stability–reversal surface.

4 Closure dependence: the escape basin

Theorems 1–3 are closure–free. The *global* fate is not.

Theorem 5 (Finite–time escape for super–linear closures)**.** *For the cubic closure $f_{\mathrm{cub}}(r) = r - r^3$ and $r_0 > 1$, there is a threshold $z_0^*(\varepsilon_0)$ such that any trajectory with $z_0 < z_0^*(\varepsilon_0)$ blows up in finite time, escaping to $r = \infty$ while still in the region $z < 0$. Empirically (bisection at* `rtol`$= 10^{-10}$*),*

$$z_0^*(\varepsilon_0) \;\approx\; -\ln\!\Big[\tfrac{-\ln \varepsilon_0 + C}{2}\Big], \qquad C \approx 3.68, \tag{8}$$

with $z_0^*(0.01) \approx -1.495$ (blow–up at $t \approx 0.38$).

The double–logarithmic form of (8) is not fitted noise: escape occurs when the linear transient amplification $\varepsilon_0 \exp(2e^{-z_0})$ first reaches $O(1)$ — i.e. when $\ln \varepsilon_0 + 2e^{-z_0} \sim$ const — which inverts precisely to (8). Thus the near–helix linear analysis *predicts its own basin boundary*, with the nonlinearity fixing only the $O(1)$ threshold. Figures 3–4 display the split and the boundary curve.

Proposition 6 (Global attraction for bounded closures)**.** *If f is bounded and satisfies (3), then no trajectory of (2) with $r_0 > 0$ escapes in finite time, and every such trajectory satisfies $r(t) \to 1$.*

Proof. Boundedness gives $|\dot r| \le \|f\|_\infty |1 - e^{-z}| \le \|f\|_\infty$, so r grows at most linearly and remains finite for all finite t: no blow–up. Since $z = z_0 + t \to +\infty$, eventually $1 - e^{-z} > 0$; on that region (3) gives $\dot r = f(r)(1 - e^{-z})$ with $f(r)(r - 1) < 0$, so $r = 1$ is globally attracting in $r > 0$ and $r(t) \to 1$. □

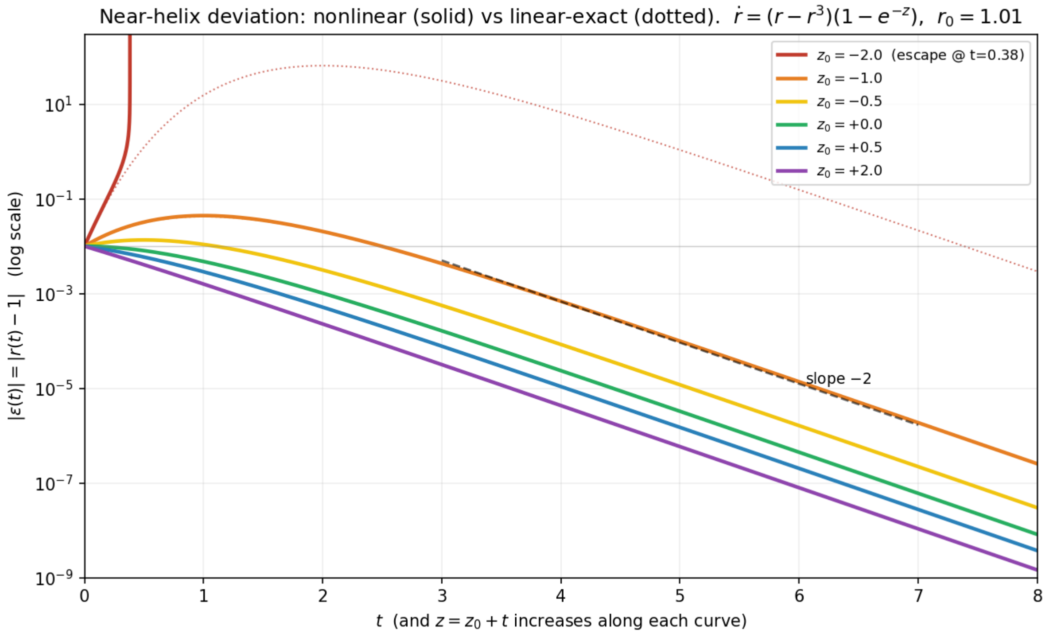

Figure 3: Cubic closure, $r_0 = 1.01$. Solid: nonlinear $|\varepsilon(t)|$; dotted: the exact linear prediction (6). For $z_0 \gtrsim z_0^*$ all curves decay at slope -2 (the exponent of Thm. 1). The $z_0 = -2$ curve escapes at $t \approx 0.38$: a finite–time event the *linear* model structurally cannot exhibit.

Corollary 7. *"Where the flow fails to converge" is a genuine, sharply predictable prediction* iff *the closure permits super–linear growth. The bounded closure $f_{\rm sat}$ realises global attraction with the identical linearisation* (5)*; e.g. at $z_0 = -5$, $\varepsilon_0 = 0.01$ it drifts to $r_{\max} \approx 24.5$ and returns to $|\varepsilon| \sim 10^{-15}$. The linearisation alone does not determine which case obtains.*

5 Bifurcation structure: a degenerate Hopf

The events of §3 concern the *real* transverse eigenvalue of the existing cycle. A distinct, genuinely Hopf event lives at the axis.

Theorem 8 (Degenerate Hopf at $r = 0$)**.** *In Cartesian coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ with $\dot\theta = \omega$, the linearisation of* (2) *at the axis equilibrium $r = 0$ is*

$$J(0) = \begin{pmatrix} \lambda(z) & -\omega \\ \omega & \lambda(z) \end{pmatrix}, \qquad \lambda(z) = 1 - e^{-z},$$

with eigenvalues $\lambda(z) \pm i\omega$ crossing the imaginary axis transversally at $z = 0$ ($\lambda'(0) = 1$). The radial normal form is $\dot r = \lambda(z)\, r - a(z)\, r^3$ with $a(z) = 1 - e^{-z}$; hence the first Lyapunov coefficient $a(z) \to 0$ at criticality. The bifurcation is therefore degenerate*: the emergent cycle radius is $\sqrt{\lambda/a} \equiv 1$, pinned rather than opening as $\sqrt{\lambda}$.*

Proof. Near $r = 0$, $\dot r = f(r)(1-e^{-z}) = r(1-e^{-z}) + O(r^3)$; differentiating (x, y) gives $J(0)$ as stated, eigenvalues $\lambda \pm i\omega$. The Hopf transversality and nonzero–frequency conditions hold for $\omega \neq 0$. The cubic term of $f_{\rm cub}$ supplies $a(z) = 1 - e^{-z}$, which coincides with the linear coefficient, so the cycle amplitude $\sqrt{\lambda/a} = 1$ for all $z > 0$ and does not emanate continuously from the origin. ∎

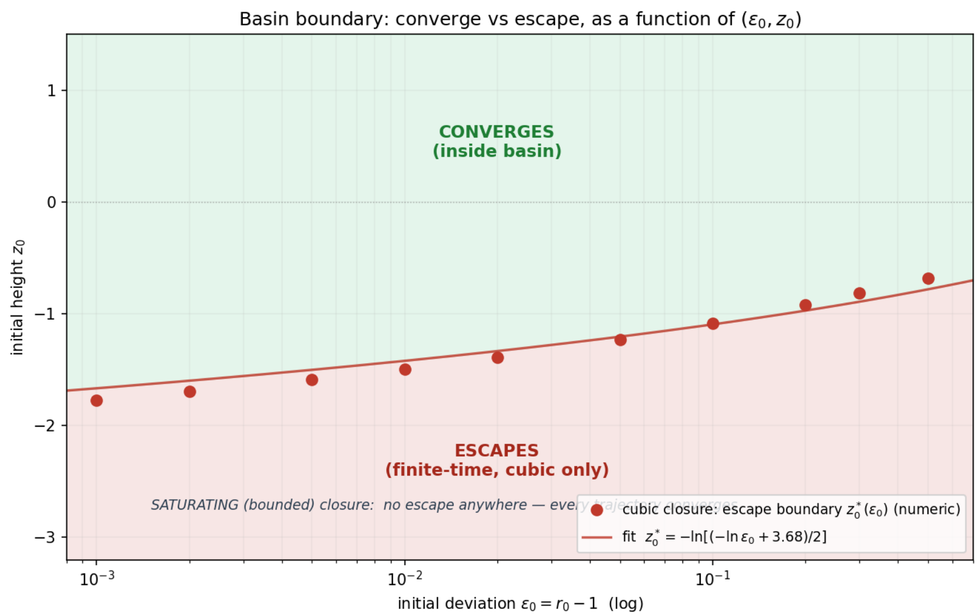

Figure 4: Basin boundary in the (ε_0, z_0) plane. Points: numerically located escape threshold for the cubic closure; curve: the double–log fit (8). Above the curve, trajectories converge; below, they escape in finite time. The bounded (saturating) closure has *no* such boundary — it converges everywhere.

Corollary 9 (Fork, bifurcation form)**.** *Retaining $\Gamma = \{r = 1\}$ (constant–radius helix) forces the degenerate Hopf ($a \equiv \lambda$, cycle pre–exists). Demanding a* generic *supercritical Hopf (fixed a, radius $\sqrt{1 - e^{-z}}$ born at $z = 0$) forces a z–dependent helix radius, breaking the constant helix. The two are mutually exclusive; see Fig. 5.*

6 A cosmological correspondence and its limits

The z–axis admits a clean *kinematic* correspondence, which we state carefully and then bound.

Proposition 10 (Kinematic de Sitter correspondence)**.** *Under the identification $z = \ln a$ (number of e–folds, a the scale factor), $\dot{z} = \dot{a}/a = H$ (Hubble rate), so $\dot{z} = 1 \Leftrightarrow H = 1 \Leftrightarrow a = e^t$ (de Sitter); and $e^{-z} = a^{-1} \propto T \propto (1 + \text{redshift})$. Moreover the standard Friedmann rate in e–folds, $H(z) = H_0\sqrt{\Omega_\Lambda + \Omega_m e^{-3z} + \Omega_r e^{-4z}}$, and the model's normalised attraction $1 - e^{-z}$ share the same asymptotic structure: a constant de Sitter attractor dressed with corrections that decay exponentially in e–folds (Fig. 6).*

This correspondence is *kinematic*, not a derivation. Three obstructions prevent its promotion to a dynamical model:

(O1) **Incomplete dictionary.** Only z is interpreted; r (pinned at 1, non–expanding) and θ have no cosmological meaning. The expanding coordinate is z alone.

(O2) **Asymptotic only.** Constant H is the $\Omega_\Lambda = 1$ future limit; it has no matter or radiation era and hence contradicts the securely measured decelerating epochs (nucleosynthesis, the acoustic peaks). A realistic history needs $\dot{z} = H(z)$ *varying.*

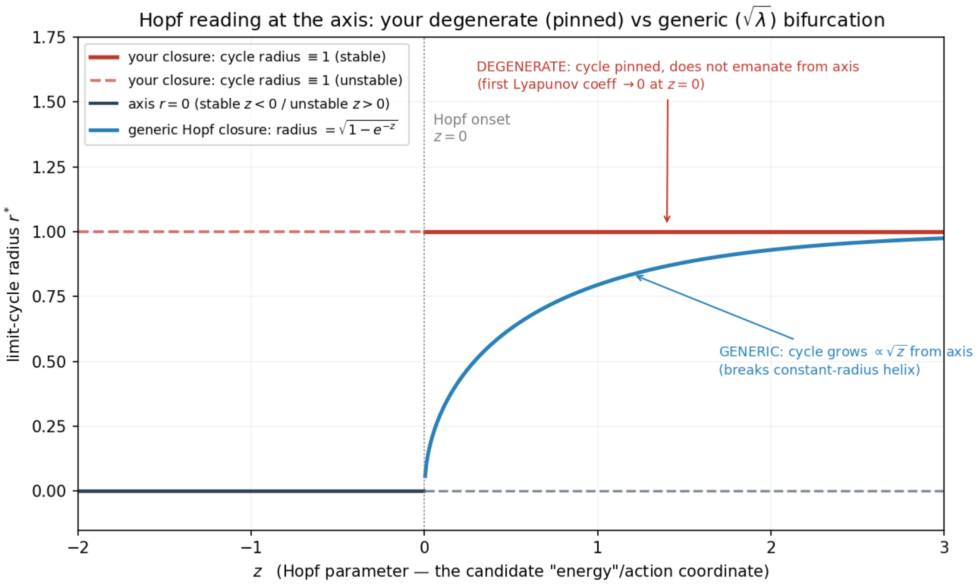

Figure 5: Bifurcation diagram at the axis. Pinned branch (red): the framework's degenerate Hopf, radius $\equiv 1$, first Lyapunov coefficient vanishing at $z = 0$. Parabolic branch (blue): the generic supercritical Hopf, $r^* = \sqrt{1 - e^{-z}}$, which breaks the constant–radius helix.

(O3) **Wrong power.** The correction $e^{-z} = a^{-1}$ corresponds to equation of state $w = -\frac{2}{3}$ (a frustrated–network component), not to matter a^{-3} or radiation a^{-4}.

7 A contact–Hamiltonian no–go

We ask whether the geometry *forces* an expansion law $H(z)$, promoting $\dot{z} = 1$ to $\dot{z} = H(z)$. The contact Hamilton equations for $\mathcal{H}(\theta, p, z)$ with $\alpha = \mathrm{d}z - p\,\mathrm{d}\theta$ read [4]

$$\dot{\theta} = \partial_p \mathcal{H}, \qquad \dot{p} = -\big(\partial_\theta \mathcal{H} + p\,\partial_z \mathcal{H}\big), \qquad \dot{z} = p\,\partial_p \mathcal{H} - \mathcal{H}. \tag{9}$$

Theorem 11 (No coexistence of attractor and de Sitter rate)**.** *Consider* (9) *with constant rotation* $\dot{\theta} \equiv 1$.

(a) Then $\mathcal{H} = p + g(\theta, z)$ for some g, and consequently $\dot{p}$ is affine in p: it contains no p^2 term. The cubic limit cycle ($\dot{p} \ni -2p^2(1 - e^{-z})$) is therefore not *generated by any such contact Hamiltonian.*

(b) In the θ–independent linear reduction $g = g(z)$, the expansion rate and transverse rate are $H(z) := \dot{z} = -g(z)$ and $c(z) = -g'(z)$, so

$$c(z) \;=\; H'(z) \qquad \textit{(locking identity).} \tag{10}$$

(c) Hence if $c(z) \to -2$ (a transverse attractor, matching $\mu_{\max}$), then $H(z) = H(z_0) + \int_{z_0}^{z} c \to -\infty$: the expansion rate is unbounded below and cannot converge to a positive constant. A transverse contact attractor and a de Sitter expansion rate are mutually exclusive on (M, α).

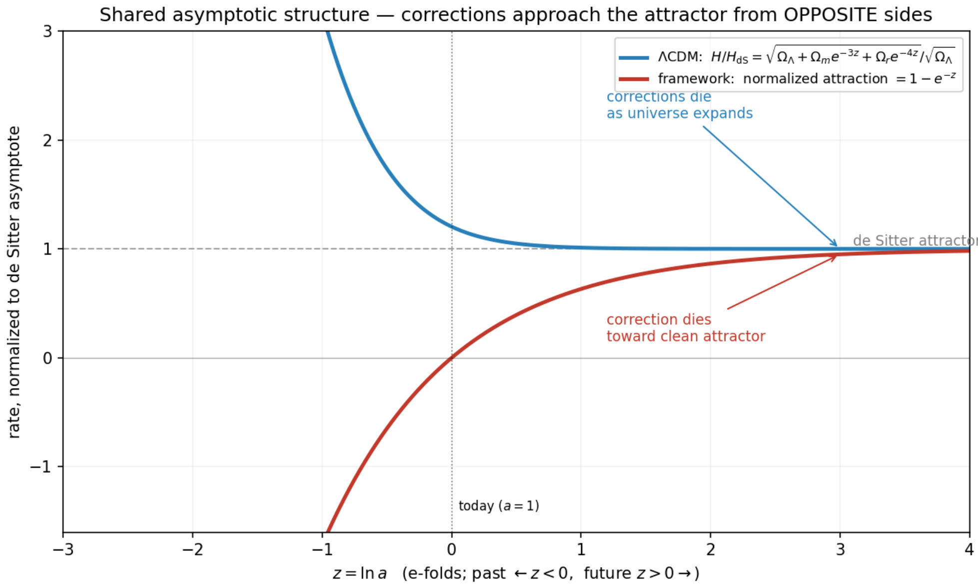

Figure 6: Shared asymptotic structure. ΛCDM approaches the de Sitter attractor from above (early universe expands faster); the model's attraction approaches it from below (early attraction weaker, repelling for $z < 0$). Same asymptote; opposite–signed, different–power corrections. Coincidence of the two would be spurious; the honest statement is structural analogy.

Proof. (a) $\partial_p \mathcal{H} = \dot\theta = 1$ forces $\mathcal{H} = p + g(\theta, z)$. Then $\dot p = -\partial_\theta g - p\,\partial_z g$, affine in p; a p^2 term is impossible. (b) With $g = g(z)$: $\dot z = p \cdot 1 - (p + g) = -g(z) =: H(z)$ and $\dot p = -g'(z)\,p$, i.e. $c(z) = -g'(z) = H'(z)$. (c) Integrate (10): $H(z) - H(z_0) = \int_{z_0}^{z} c$. If $c \to -2$ the integral diverges to $-\infty$, so $H \to -\infty$; in particular $H \not\to L$ for any finite $L > 0$. □

Remark 12. Every contact–structure–preserving flow *is* a contact Hamiltonian flow; thus Theorem 11(a) says the stated dynamics, under constant rotation, do not preserve the advertised contact structure — the form α is, with respect to the flow, decorative unless $\dot\theta$ is allowed to vary with r. The single available escape (variable rotation $\dot\theta = \omega(r)$) relaxes (3)'s constant period to a cycle–only statement and owes its own derivation.

8 Discussion

The model is a faithful *relaxation* system: a dissipative flow whose transverse contraction ($\mu = -2$) and whose z–advance are, under the contact structure, rigidly coupled through (10). That coupling is exactly what makes it a poor *expansion* model — Theorem 11 is a structural obstruction, not a gap to be closed. The productive readings are therefore (i) the swept–bifurcation transient of §3, (ii) the closure–diagnostic of §4 (super–linear vs. bounded decides global fate), and (iii) the degenerate Hopf of §5 as a non–generic normal form worth cataloguing. The cosmological correspondence of §6 is genuine but strictly kinematic; its three obstructions and the no–go together delimit precisely what the geometry can and cannot claim.

A Exercises

Exercise 1 (Closed form)**.** Verify by direct differentiation that (6) solves (5). *Hint:* compute $\frac{\mathrm{d}}{\mathrm{d}t}\ln|\varepsilon|$.

Exercise 2 (Exponent)**.** From (6), show $\mu = -2$ and identify the exact bounded correction to $\ln|\varepsilon|$. Explain, in one sentence, why an e^{-z} term with z growing *sub*linearly (say $z = \sqrt{t}$) would *not* leave μ invariant.

Exercise 3 (Transient peak)**.** For $z_0 < 0$ show the transient $|\varepsilon|$ peaks at $z = 0$ (i.e. $t = -z_0$) with amplification $|\varepsilon(t^*)|/|\varepsilon_0| = \exp\big(2e^{-z_0} - 2|z_0| - 2\big)$, and give its leading behaviour as $z_0 \to -\infty$.

Exercise 4 (Neutral line)**.** Prove Theorem 3. Why is $z = 0$ *not* a saddle–node, transcritical, or pitchfork bifurcation of (2)? *Hint:* examine what the whole vector field does at $z = 0$.

Exercise 5 (Finite–time escape)**.** For the cubic closure freeze $z < 0$ so that $1 - e^{-z} =: -\kappa < 0$. Show $\dot{r} = -\kappa(r - r^3)$ has solutions leaving every bounded set in finite time for $r_0 > 1$, and estimate the blow–up time. Explain why the swept problem ($z = z_0 + t$) escapes only when z_0 is sufficiently negative.

Exercise 6 (Global attraction)**.** Prove Proposition 6. Where exactly does boundedness of f enter, and where does condition (3) enter?

Exercise 7 (Hopf eigenvalues)**.** Derive $J(0)$ in Theorem 8 and its eigenvalues. Verify the Hopf transversality condition and state where nonzero frequency is used.

Exercise 8 (Generic vs. degenerate)**.** For $\dot{r} = \lambda r - ar^3$ with *constant* $a > 0$, show the stable cycle radius is $\sqrt{\lambda/a}$. Contrast with $a = \lambda = 1 - e^{-z}$ and explain in one sentence why the framework's helix has fixed radius.

Exercise 9 (Cosmological kinematics)**.** Show $z = \ln a \Rightarrow \dot{z} = H$. Identify the de Sitter case and compute the equation of state w for which $\rho \propto a^{-1}$ (i.e. the "power" of the e^{-z} correction). Which standard component, if any, does it match?

Exercise 10 (Contact no–go)**.** Starting from (9), impose $\dot{\theta} \equiv 1$ and derive $\mathcal{H} = p + g(\theta, z)$. Prove the locking identity (10) in the θ–independent case and deduce Theorem 11(c). Finally, sketch how a variable rotation $\dot{\theta} = \omega(r)$ evades part (a).

B Solutions (sketches)

1. $\frac{\mathrm{d}}{\mathrm{d}t}\ln|\varepsilon| = -2 + 2e^{-z_0} \cdot e^{-t} = 2(e^{-(z_0+t)} - 1)$, matching (5).

2. $\ln|\varepsilon/\varepsilon_0| = -2t + 2e^{-z_0}(1 - e^{-t})$; the bounded correction is $2e^{-z_0}(1 - e^{-t}) \to 2e^{-z_0}$, so $\mu = -2$. With $z = \sqrt{t}$, $\int^\infty e^{-\sqrt{s}}\,\mathrm{d}s = \infty$ diverges more slowly than t but the rate term -1 is unaffected; the point is that any $\int_0^t e^{-z(s)}\,\mathrm{d}s$ growing linearly in t *would* shift μ — integrability is what protects it. (For $z = \sqrt{t}$, $\frac{1}{t}\int_0^t e^{-\sqrt{s}}\,\mathrm{d}s \to 0$, so $\mu = -2$ still; the invariance fails only if e^{-z} has nonzero Cesàro mean, e.g. z bounded.)

3. $\ln|\varepsilon|$ is maximised where $\rho(z) = 0$, i.e. $z = 0$, $t = -z_0$. Substitute into (6): exponent $-2(-z_0) + 2e^{-z_0}(1 - e^{z_0}) = 2z_0 + 2e^{-z_0} - 2$; hence amplification $\exp(2e^{-z_0} - 2|z_0| - 2)$ (using $|z_0| = -z_0$). As $z_0 \to -\infty$ it grows like $\exp(2e^{|z_0|})$.

4. Sign of ρ is that of $-z$; $z = 0$ is the unique zero and $\rho'(0) = -2 \neq 0$ for the *rate*, but the full field $\dot{r} = f(r)(1 - e^{-z})$ vanishes identically at $z = 0$ (every r is momentarily fixed), so no fixed point is created, destroyed, or exchanged — disqualifying the codimension–one normal forms. It is a neutral stability–reversal locus.

5. $\dot{r} = -\kappa(r - r^3) = \kappa r^3(1 - r^{-2})$; for $r_0 > 1$, $\dot{r} > 0$ and $\dot{r} \gtrsim \kappa r^3/2$ for $r \geq \sqrt{2}$, whose solution $\sim (r_0^{-2} - \kappa t)^{-1/2}$ blows up by $t_* \lesssim r_0^{-2}/\kappa$. In the swept problem $\kappa = e^{-z} - 1$ shrinks as $z \to 0^-$; only sufficiently negative z_0 gives a window long/strong enough to reach blow–up before z crosses 0.

6. See Prop. 6: boundedness prevents finite–time blow–up (bounds $|\dot{r}|$); (3) gives the sign of f that makes $r = 1$ globally attracting once $1 - e^{-z} > 0$.

7. $\dot{x} = \lambda x - \omega y$, $\dot{y} = \omega x + \lambda y$ to first order; eigenvalues $\lambda \pm i\omega$; Re $= \lambda(z)$, $\lambda'(0) = e^0 = 1 \neq 0$ (transversal); $\omega \neq 0$ ensures genuine rotation (nonzero imaginary part).

8. Setting $\dot{r} = 0$: $r^2 = \lambda/a$. For constant a, $r^* = \sqrt{\lambda/a} \propto \sqrt{\lambda}$. For $a = \lambda$, $r^* = 1$ independent of λ: the amplitude decouples from the parameter, the degenerate case.

9. $\dot{z} = \frac{\mathrm{d}}{\mathrm{d}t} \ln a = \dot{a}/a = H$; $H \equiv 1 \Rightarrow a = e^t$ (de Sitter). $\rho \propto a^{-3(1+w)} = a^{-1} \Rightarrow w = -\frac{2}{3}$: a domain–wall/frustrated–network component, matching neither matter ($w = 0$) nor radiation ($w = \frac{1}{3}$).

10. $\partial_p \mathcal{H} = 1 \Rightarrow \mathcal{H} = p + g(\theta, z)$; θ–independent $\Rightarrow \dot{z} = -g$, $\dot{p} = -g'p$, so $c = -g' = H'$. If $c \to -2$, $H = \int c \to -\infty$, excluding a positive constant limit. Variable rotation $\dot{\theta} = \omega(r)$ makes $\partial_p \mathcal{H}$ depend on p, admitting p^2 in $\dot{p}$ and evading part (a) — at the cost of constant period.

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