Affine orthogonal quiver varieties at δ and 2δ, and their contact links

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Abstract

Pinot recently computed the quiver varieties $\mathfrak{M}_\delta$ and $\mathfrak{M}_{2\delta}$ of the orthogonal quivers whose underlying graph is an affine Dynkin diagram, and identified Kleinian surface singularities among them [14]. We make three observations and one construction. (i) Assuming that the closed immersion of Li [9, Prop. 9.2.1] applies to Pinot's unframed signed setting (we have not checked this), whenever $\mathfrak{M}_\delta$ is a surface the involution acts trivially and $\mathfrak{M}_\delta$ is the classical Kleinian surface of the underlying affine diagram. (ii) Consequently the two $\widetilde{D}$ rows of [14, Thm. 1.2] are of type D_{2n-2} and D_{2n-1}, not D_{n+1}; for $(\widetilde{D}_{2n-1}, a)$ the defining relation is $z^2 = z(-x)^{n-1} - xy^2$. We confirm both by an independent numerical computation of the invariant ring. (iii) In every row at 2δ the group Γ contains the classical group of the underlying graph with index 2 (or 1); we conjecture that this quotient structure arises naturally from the quiver data, and prove the corresponding statement for the coordinate rings. Finally, each surface is a cone whose link is a contact 3-manifold S^3/Γ; we show that the symmetric representation space is quaternionic, so the link is the 3-Sasakian (Kempf–Ness) quotient of the unit sphere, and we tabulate the Reeb (Hopf) Seifert fibrations.

How to read this version. Every statement carries one tag: [proved here], [cited], [numerically checked] (a script in the bundle; evidence, not proof), [kernel-checked arithmetic] (Lean 4, arithmetic only), or [open]. **Verification bundle** (Python and Lean, with a README): click here to open the attached zip (attached to this PDF), or download it from https://totogt.github.io/geometry/collab/pinot-orthogonal-quivers/. This text was prepared with the assistance of Claude (Anthropic); the author takes responsibility for it, and it has not yet been checked by V. Pinot.

1 Introduction

Nakajima quiver varieties of affine Dynkin type recover Kronheimer's ALE spaces and the Kleinian singularities $\mathbb{C}^2/\Gamma$ [7, 12]; the ADE labels go back to Gabriel's classification of quivers of finite type [5]. Pinot studies the analogous quotients for *orthogonal quivers* [14]: quivers Q with a contravariant involution τ and a sign s, whose symmetric representations $\mathrm{SRep}(\overline{Q}, \mathrm{d})$ carry an action of $G_{\mathrm{d}}^b = \prod O(\mathrm{d}_i) \times \prod GL(\mathrm{d}_j)$, and

$$\mathfrak{M}_{\mathrm{d}} = \mu^{-1}(0)/\!/G_{\mathrm{d}}^b.$$

His Theorem 1.2 lists $\mathfrak{M}_\delta$ and $\mathfrak{M}_{2\delta}$ for ten infinite families. This note re-reads that table from two angles: the comparison with the classical quiver variety of the same graph (§§3–4), and the contact geometry of the link of each surface (§5).

Results.

- **Theorem A** (§3). If $\mathfrak{M}_\delta$ is a surface, then $\mathfrak{M}_\delta \cong \mathfrak{M}_\delta^{\mathrm{cl}}$, the classical Kleinian surface of the underlying affine graph, and the involution acts trivially on it. Conditional only on the applicability of Li's closed immersion to Pinot's unframed setting.
- **Corollary B** (§3). The $\widetilde{D}$ rows read D_{2n-2} and D_{2n-1}. The $(\widetilde{D}_{2n-2}, v)$ equation of [14] is correct; the $(\widetilde{D}_{2n-1}, a)$ relation is $z^2 = z(-x)^{n-1} - xy^2$. Confirmed numerically for $n = 3, 4, 5$ by computing the invariant ring directly.
- **Conjecture C** (§4). Each 2δ surface is $\mathfrak{M}_\delta^{\mathrm{cl}}/\iota$ for an involution ι constructed naturally from the quiver data (the abstract quotient statement follows once the invariant-ring identifications are established); equivalently $\Gamma_{2\delta} \supset \Gamma_{\mathrm{cl}}$ with index ≤ 2, and the classical link double-covers the 2δ link (index 2 rows; the $(-,-)$ row has index 1). The ring-level statement is proved.
- **Theorem D** (§5). $\mathrm{SRep}(\overline{Q}, \mathrm{d})$ is a quaternionic subspace; the link is the Kempf–Ness / 3-Sasakian quotient of the unit sphere; the Reeb flow is the Hopf flow, with the Seifert data of Table 2.

2 Setting

We use the notation of [14] throughout, with the standard signed space ($J_i = \mathrm{id}$ in suitable bases), so that the involution on representations is $\sigma(v)_a = s(a)\, v_{\tau a}^T$ and $\mathrm{SRep} = \mathrm{Rep}^\sigma$ [14, Def. 2.3, 2.5]. The moment map is the restriction of the quiver moment map, $\mu(\mathrm{SRep}) \subset \mathfrak{g}^b$ [14, Cor. 2.13]. The invariant ring is generated by traces of cycles [14, Thm. 3.9], [8]. The scaling $t \cdot v$ gives $\mathrm{tr}(\Lambda^k C)$ the weight $k\,\ell(C)$, so every $\mathfrak{M}_{\mathrm{d}}$ is a cone [14, §3.5.1].

The comparison map. Let $\mathfrak{M}_{\mathrm{d}}^{\mathrm{cl}} = \mu^{-1}(0)/\!/G_{\mathrm{d}}$ be the classical quiver variety of the same quiver. By universality of categorical quotients there is a morphism $\iota : \mathfrak{M}_{\mathrm{d}} \to (\mathfrak{M}_{\mathrm{d}}^{\mathrm{cl}})^{[\sigma]}$. Remark 2.15 of [14] states that it is an isomorphism, citing [9, 13]. The sources give less: Li proves that ι is a *closed immersion* for an arbitrary graph [9, Prop. 9.2.1], calls the isomorphism for ι conjectural for any graph and expected in type A_n [9, Rem. 9.2.3]; his Proposition 9.2.2 proves an isomorphism only for an enlarged map $\tilde{\iota}$ [9, Prop. 9.2.2]; Nakajima treats only smooth σ-quiver varieties (generic ζ) [13]. Li works with framed varieties (v, w) and prescribed forms; we *assume* throughout that his Proposition 9.2.1 applies to Pinot's unframed signed setting (Question 2). We use only the closed immersion.

3 The surfaces at δ

Theorem 3.1. *Assume Li's closed immersion $\iota : \mathfrak{M}_\delta \hookrightarrow \mathfrak{M}_\delta^{\mathrm{cl}}$ holds in Pinot's setting, δ being the minimal imaginary root of the underlying affine graph. If $\mathfrak{M}_\delta$ is a reduced irreducible surface (as in all non-trivial rows of [14, Thm. 1.2]), then ι is an isomorphism and $[\sigma]$ acts*

trivially on $\mathfrak{M}_\delta^{\mathrm{cl}}$. In particular $\mathfrak{M}_\delta \cong \mathbb{C}^2/\Gamma$ with Γ the McKay group of the underlying affine graph.

Proof. $\mathfrak{M}_\delta^{\mathrm{cl}} \cong \mathbb{C}^2/\Gamma$ is an irreducible surface [7, 12]. A closed immersion of a reduced irreducible surface into an irreducible surface is surjective, hence an isomorphism onto it. Since its image lies in the fixed locus of $[\sigma]$, that fixed locus is everything. [proved here] (conditional on the hypothesis). □

Corollary 3.2. *The non-trivial δ rows are of type A_{2n-2} for $\widetilde{A}_{2n-2}$, A_{2n-1} for $\widetilde{A}_{2n-1}$, D_{2n-2} for $(\widetilde{D}_{2n-2}, v)$ and D_{2n-1} for $(\widetilde{D}_{2n-1}, a)$. The last two replace the label D_{n+1} printed in [14, Thm. 1.2]; they agree with it only for $n = 3$ in the first case, and never in the second.*

Corollary 3.2 depends on the hypothesis of Theorem 3.1. The next two results check it independently.

Proposition 3.3 (weighted monomials)**.** *In the $\widetilde{D}$ rows let $x = \det C_3$, $y = \operatorname{tr}(C_1C_2)$, $z = \operatorname{tr}(C_1C_2C_3)$, with $L = 2n-5$ for $(\widetilde{D}_{2n-2}, v)$ and $L = 2n-4$ for $(\widetilde{D}_{2n-1}, a)$ [14, p. 18]. Their weights are $(4, 2L+2, 2L+4)$, and a relation homogeneous with z^2 has degree $4L+8$. The only monomials of that degree are:*

- *$(\widetilde{D}_{2n-2}, v)$, $n \geq 4$: $z^2,\ xy^2,\ x^{n-1}y,\ x^{2n-3}$. The relation $z^2 + \beta xy^2 + \alpha x^{n-1}y + \gamma x^{2n-3}$ is of type D_{2n-2} if and only if $\beta \neq 0$ and $\gamma \neq \alpha^2/4\beta$. For $n = 3$ the weights are $(4,4,6)$, so y^3 also has degree 12: the relation is $z^2 + c(x,y)$ with c a binary cubic, of type D_4 if and only if c has three distinct linear factors.*
- *$(\widetilde{D}_{2n-1}, a)$: $z^2,\ x^{n-1}z,\ xy^2,\ x^{2n-2}$. The relation $z^2 + \alpha x^{n-1}z + \beta xy^2 + \gamma x^{2n-2}$ is of type D_{2n-1} if and only if $\beta \neq 0$ and $\gamma \neq \alpha^2/4$. The term $x^{n-1}y$ printed in [14, Lemma 3.27] has degree $4L+6$ and cannot occur.*

Proof. Solve $4a + (2L+2)b + (2L+4)c = 4L+8$ in non-negative integers. Then complete the square: in y for the first row, in z for the second. The result is the normal form $z^2 + xy^2 + c\,x^{m-1}$ of D_m with $m = 2n-2$, resp. $m = 2n-1$, and $c \neq 0$ exactly under the stated conditions. Equivalently, by the Milnor–Orlik formula $\mu = \prod(d/w_i - 1) = L+3$ [11]. [proved here] [kernel-checked arithmetic] □

Proposition 3.4 (direct computation)**.** *For $n = 3, 4, 5$ we sampled 80 random points of $\mu^{-1}(0) \cap \mathrm{SRep}(\overline{Q}, \delta)$ by Gauss–Newton and found all linear relations among the monomials of weight $\leq 4L+8$ in x, y, z. In each case there is exactly one relation (relative singular value $\leq 2 \cdot 10^{-8}$, against ≥ 0.1 for the next), with integer coefficients:*

$$(\widetilde{D}_{2n-2}, v):\ z^2 = -y(-x)^{n-1} - xy^2, \qquad (\widetilde{D}_{2n-1}, a):\ z^2 = z(-x)^{n-1} - xy^2.$$

The first is the equation of [14, Lemma 3.28]. The second is [14, Lemma 3.27] with y replaced by z in the first term: the step $\operatorname{tr}(C_1C_2^2C_3^3) = \operatorname{tr}(C_1C_2C_3)(-\det C_3)^{n-1}$ appears to have been written with $\operatorname{tr}(C_1C_2)$. In both rows the non-degeneracy conditions of Proposition 3.3 hold ($\beta = 1$, discriminant $-1/4$; for $n = 3$ in the first row the cubic is $xy(x+y)$, with three distinct factors). For sign -1 in $(\widetilde{D}_{2n-1}, a)$, all invariants vanish, in agreement with [14]. [numerically checked]

quiver	sign	d	equation	type	Γ	$\lvert\Gamma\rvert$
$(\widetilde{A}_{2n-2}, v\text{-}a)$	$+1$	δ	$xy = z^{2n-1}$	A_{2n-2}	$\mathbb{Z}_{2n-1}$	$2n-1$
	-1	2δ	$xy = z^{4n-2}$	A_{4n-3}	$\mathbb{Z}_{4n-2}$	$4n-2$
$(\widetilde{A}_{2n-1}, a\text{-}a)$	$(+,+)$	δ	$xy = z^{2n}$	A_{2n-1}	$\mathbb{Z}_{2n}$	$2n$
	$(+,-)$	2δ	$xy = z^{4n}$	A_{4n-1}	$\mathbb{Z}_{4n}$	$4n$
	$(-,-)$	2δ	$xy = z^{2n}$	A_{2n-1}	$\mathbb{Z}_{2n}$	$2n$
$(\widetilde{A}_{2n-1}, v\text{-}v)$		δ	$xy = z^{2n}$	A_{2n-1}	$\mathbb{Z}_{2n}$	$2n$
$(\widetilde{A}_{2n-1}, c)$		2δ	$x^{n+1} - y^2x - z^2 = 0$	D_{n+2}	BD_{4n}	$4n$
$(\widetilde{D}_{2n-2}, v)$		δ	$z^2 = -y(-x)^{n-1} - xy^2$	$\mathbf{D_{2n-2}}$	$\mathrm{BD}_{8(n-2)}$	$8n-16$
$(\widetilde{D}_{2n-1}, a)$	$+1$	δ	$\mathbf{z^2 = z(-x)^{n-1} - xy^2}$	$\mathbf{D_{2n-1}}$	$\mathrm{BD}_{4(2n-3)}$	$8n-12$

Table 1: The non-trivial rows of [14, Thm. 1.2], with the corrections in bold. BD_{4k} is the binary dihedral group of order $4k$ (for $n = 1$ in the $(\widetilde{A}_{2n-1}, c)$ row, $D_3 = A_3$). All other rows are reduced points.

4 The surfaces at 2δ

At 2δ the classical variety is 4-dimensional, and the orthogonal surfaces are new. Comparing each $\Gamma_{2\delta}$ with the classical group Γ_{cl} of the same graph:

2δ row	$\Gamma_{2\delta}$	Γ_{cl}	index
$(\widetilde{A}_{2n-2}, v\text{-}a)$, -1	$\mathbb{Z}_{4n-2}$	$\mathbb{Z}_{2n-1}$	2
$(\widetilde{A}_{2n-1}, a\text{-}a)$, $(+,-)$	$\mathbb{Z}_{4n}$	$\mathbb{Z}_{2n}$	2
$(\widetilde{A}_{2n-1}, c)$	BD_{4n}	$\mathbb{Z}_{2n}$	2
$(\widetilde{A}_{2n-1}, a\text{-}a)$, $(-,-)$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_{2n}$	1

The two index-2 overgroups of $\mathbb{Z}_{2n}$ in $SU(2)$ that occur here, $\mathbb{Z}_{4n}$ and BD_{4n}, both appear.

Conjecture 4.1. Each 2δ surface is isomorphic to $\mathfrak{M}_\delta^{\mathrm{cl}}/\iota$ for an involution ι of the classical surface that arises naturally from the quiver construction, trivial exactly in the $(-,-)$ row. (The abstract isomorphism with a quotient by an involution follows from $\Gamma_{\mathrm{cl}} \subset \Gamma_{2\delta}$ once the invariant rings are identified; what is conjectural is that the involution and quotient map come from the quiver data.)

Proposition 4.2. *Write $\mathbb{C}^2/\mathbb{Z}_m = \{XY = Z^m\}$ with $X = u^m$, $Y = v^m$, $Z = uv$.*

(a) The generator of $\mathbb{Z}_{2m} \supset \mathbb{Z}_m$ induces $\iota_1 : (X, Y, Z) \mapsto (-X, -Y, Z)$. Its invariants X^2, Y^2, Z satisfy $X^2Y^2 = Z^{2m}$, the surface A_{2m-1}.

(b) For $m = 2n$, $j \in \mathrm{BD}_{4n}$ induces $\iota_2 : (X, Y, Z) \mapsto (Y, X, -Z)$. Its invariants $s = X + Y$, $w = Z^2$, $t = Z(X - Y)$ satisfy $t^2 = ws^2 - 4w^{n+1}$, the surface D_{n+2}.

With $m = 2n - 1$, resp. $m = 2n$, these reproduce the three index-2 rows above exactly.

Proof. Direct substitution; the invariant rings are the classical Klein invariants of $\mathbb{Z}_{2m}$ and BD_{4n}. [proved here] [numerically checked] (symbolic, $m \le 11$). □

Pinot's own generators fit this picture. In the $(+,-)$ row the 2δ coordinates are $\det C_1, \det C_2$ and $\operatorname{tr} C_3$, where the classical δ coordinates are $\operatorname{tr} C_1, \operatorname{tr} C_2, \operatorname{tr} C_3$. The weights double for the first two, just as $X \mapsto X^2$, $Y \mapsto Y^2$ in (a). The remaining step is to construct ι from the quiver data. [open]

5 The contact links

5.1 Cones and weights

Lemma 5.1. *In every row of Table 1, Pinot's path-length weights of the three generators equal the degrees of the corresponding Klein invariants in $\mathbb{C}[u,v]^\Gamma$: $(m,m,2)$ for $\mathbb{Z}_m$ and $(4,2k,2k+2)$ for BD_{4k}. Hence the identification $\mathfrak{M}_\mathrm{d} \cong \mathbb{C}^2/\Gamma$ can be chosen graded, and the unit circle of the scaling $\mathbb{C}^*$ acts on $\mathbb{C}^2/\Gamma$ by scalars.*

Proof. At δ for $\widetilde{A}_{m-1}$: $\operatorname{tr} C_1, \operatorname{tr} C_2$ have weight m, $\operatorname{tr} C_3$ weight 2. At 2δ, determinants double the cycle length: $(4n-2, 4n-2, 2)$ and $(4n, 4n, 2)$. In the $(\widetilde{A}_{2n-1}, c)$ row the weights are $(4, 2n, 2n+2)$, and in the $\widetilde{D}$ rows $(4, 2L+2, 2L+4)$ with $k = L+1$. The normal-form changes of variables used above preserve the grading. [proved here] □

The link $L_Q = (\mathfrak{M}_\mathrm{d} \setminus 0)/\mathbb{R}_{>0} \cong S^3/\Gamma$ carries its canonical contact structure: the complex tangencies, unique up to contactomorphism [4], which is the quotient of the standard structure of S^3 since $\Gamma \subset SU(2)$. It is universally tight and Milnor fillable. [cited]

5.2 A real structure

A real contact form needs a real exact symplectic form. Pinot's ω and μ are holomorphic, so we use the flat hyperkähler structure of $T^*\mathrm{Rep}(Q, \mathrm{d})$: $I = i$ and $J(x,y) = (-y^\dagger, x^\dagger)$ arrow by arrow.

Lemma 5.2. *In the standard signed space, $\sigma \circ J = J \circ \sigma$ and σ is unitary. Hence* $\mathrm{SRep}(\overline{Q}, \mathrm{d})$ *is a quaternionic subspace. The compact group $K^b = G^b \cap \prod U(\mathrm{d}_i) = \prod O(\mathrm{d}_i, \mathbb{R}) \times \prod U(\mathrm{d}_j)$ preserves I, J, K, and both moment maps send* SRep *into $\mathfrak{k}^b$.*

Proof. $\sigma(Jv)_a = s(a)(-v^\dagger_{(\tau a)^*})^T = -s(a)\overline{v_{(\tau a)^*}} = J(\sigma v)_a$, and similarly for a^*, using $s(a^*) = s(a)$ and $\tau(a^*) = \tau(a)^*$. σ permutes matrix entries up to sign and transposition, so it preserves the Hermitian norm. For $\mu_\mathbb{C}$ the last claim is [14, Prop. 2.11]; for $\mu_\mathbb{R}$ the same computation applies. [proved here] [numerically checked] □

Theorem 5.3. *There is a $\mathbb{C}^*$-equivariant homeomorphism $\big(\mu_\mathbb{R}^{-1}(0) \cap \mu_\mathbb{C}^{-1}(0) \cap \mathrm{SRep}\big)/K^b \cong \mathfrak{M}_\mathrm{d}$. Intersecting with the unit sphere gives*

$$L_Q \cong \big(\mu_\mathbb{R}^{-1}(0) \cap \mu_\mathbb{C}^{-1}(0) \cap S(\mathrm{SRep})\big)/K^b,$$

the 3-Sasakian quotient of the unit sphere in the quaternionic space SRep *[2, 1]. The smooth structure of L_Q comes from Lemma 5.1. The reduction need not be locally free on the zero set.*

Proof. $\mu_\mathbb{C}^{-1}(0) \cap \mathrm{SRep}$ is a closed G^b-stable subvariety of a unitary representation of $G^b = (K^b)_\mathbb{C}$. Kempf–Ness [6] identifies its categorical quotient with the K^b-quotient of the zero set of the K^b-moment map, which is $\mu_\mathbb{R}$ by Lemma 5.2. Both sides are cones, so the identification restricts to links. [proved here] (standard) □

5.3 Reeb flow

By Lemma 5.1, the Reeb flow of the contact form induced by the flat metric is the Hopf flow $x \mapsto e^{i\theta}x$ on S^3/Γ.

Proposition 5.4. *The Hopf flow on S^3/Γ is a Seifert fibration with the following data. The generic orbit has period 2π if $-1 \notin \Gamma$ and π otherwise. The exceptional fibre over a cone point of order a has period (generic)$/a$.*

type	Γ	L_Q	base orbifold	generic period
A_{m-1}, m odd	$\mathbb{Z}_m$	$L(m, m-1)$	$S^2(m, m)$	2π
A_{m-1}, m even	$\mathbb{Z}_m$	$L(m, m-1)$	$S^2(m/2, m/2)$	π
D_{k+2}	BD_{4k}	prism manifold	$S^2(2, 2, k)$	π

Table 2: Seifert data of the Reeb flow: base orbifold and multiplicities. The Seifert invariants β_i and the Euler number are not computed here.

Proof. The stabiliser of x is $\{\theta : e^{i\theta}x \in \Gamma x\}$. For generic x only the scalars $\pm 1 \in \Gamma$ contribute. The base is $\mathbb{C}P^1/\bar{\Gamma}$, with $\bar{\Gamma} \subset SO(3)$ cyclic of order m or $m/2$, or dihedral of order $2k$. Its cone points are the points of $\mathbb{C}P^1$ with non-trivial stabiliser in $\bar{\Gamma}$. The identity $\chi^{\mathrm{orb}} = 2/|\bar{\Gamma}|$ checks the data. [proved here] [numerically checked] [kernel-checked arithmetic] □

Under Conjecture 4.1, $S^3/\Gamma_{\mathrm{cl}} \to S^3/\Gamma_{2\delta}$ is a double covering of contact manifolds commuting with the Reeb flows when the index is 2 (and the identity when the index is 1): the link at 2δ is then a $\mathbb{Z}_2$-quotient of the classical link.

5.4 Fillings

The classical minimal resolution of $\mathbb{C}^2/\Gamma$ (Kronheimer's $\zeta \neq 0$ quotient) is a strong symplectic (Kähler) filling of L_Q. It is not Stein, since it contains the exceptional (-2)-curves. For ADE singularities it is diffeomorphic to the Milnor fibre [3], which is a Stein filling. For lens spaces with this contact structure all fillings are classified by Lisca [10]. Whether Pinot's orthogonal quotient at $\zeta \neq 0$ gives a resolution is open: only the GL factors of G^b admit characters, and the involution sends ζ to $-\zeta$ before a Weyl correction [13]. [cited] [open]

6 Questions for V. Pinot

Question 1. Do you confirm the labels D_{2n-2}, D_{2n-1} and the relation $z^2 = z(-x)^{n-1} - xy^2$ of Corollary 3.2 and Proposition 3.4?

Question 2. Does Li's Proposition 9.2.1 apply to your unframed signed setting (for instance with $w = 0$, or with an auxiliary framing vertex)? Theorem 3.1 uses only this closed immersion.

Question 3. Do you have a source for the isomorphism of Remark 2.15 at $\zeta = 0$ beyond type A_n with alternating signs?

Question 4. Do the 2δ surfaces arise as quotients of the classical surface by an involution (Conjecture 4.1)? Is there a natural candidate for ι in terms of the quiver?

Question 5. Do you plan to treat the case of your Remark 1.3, $(\widetilde{D}_{2n-1}, a)$ with sign -1 at 2δ? Under Conjecture 4.1 one would expect an overgroup of $\mathrm{BD}_{4(2n-3)}$ of index at most 2.

A Verification bundle

Attached zip; also at `https://totogt.github.io/geometry/collab/pinot-orthogonal-quivers/`. Python 3 with `numpy`, `scipy`, `sympy`; Lean 4 core (no Mathlib).

- `verify_Dtilde_relation_numeric.py`: Proposition 3.4; saved output included.
- `verify_ade_types.py`: Milnor numbers of the $\widetilde{D}$-row polynomials; homogeneity defect of the printed relation.
- `verify_2delta_index2.py`: Proposition 4.2.
- `verify_quaternionic_SRep.py`: Lemma 5.2.
- `verify_seifert.py`: Table 2 and the group orders.
- `DM3Quivers_Check.lean`: weight degrees, the Milnor–Orlik numerator identity $(d-w_x)(d-w_y)(d-w_z) = (L+3)w_xw_yw_z$, group orders, and the orbifold identities. It does not check Remark 2.15 or any geometric statement.

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